PAPER CODE NO.
MATH295

THE UNIVERSITY
of LIVERPOOL

SEPTEMBER 2002 EXAMINATIONS

Bachelor of Engineering: Year 2
Master of Engineering: Year 2

ENGINEERING ANALYSIS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FOUR questions.
Only the best FOUR questions will be counted.

This examination contributes 80% towards the final mark.
The balance comes from the marks scored in the two class tests.

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1. (a) Given

$$z(x,y) = \frac{x^3y^3}{x^3+y^3},$$

show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z.$$

[10 marks]

(b) Expand the function

$$f(x,y) = x^2y + \sin(xy)$$

as a Taylor series about the point $(x,y) = \left(0, \frac{\pi}{2}\right)$, up to and including second order terms.

[12 marks]

Express your solution in its simplest form.

[3 marks]

2. (a) State the conditions for a point (x_0, y_0) to be a critical point of the function $f(x, y)$.

[2 marks]

Find, if any, the maxima, minima and saddle points of the function

$$f(x, y) = x^2 + 2xy.$$

[9 marks]

(b) Find the gradient of the function

$$f(x, y, z) = z \ \ln(x^3 + y^3).$$

[5 marks]

(c) Find the unit vector in the direction of the vector

$$\mathbf{t} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

[4 marks]

(d) Using your results for (b) and (c) deduce that the rate at which $f(x, y, z)$ changes with respect to distance in the direction of $\mathbf{t}$ at the point $P(0, 1, 1)$ is -2.

[5 marks]

3. (a) State the formula for the Fourier series of the function

$$f(x), \quad x \in (0, 2\pi),$$

of period 2π and the formulae for the Euler -Fourier coefficients

$$a_0, a_n, b_n \quad (n = 1, 2, \ldots).$$

[4 marks]

(b) Given that

$$f(x) = \begin{cases} -x, & 0 < x < \pi, \\ 0, & \pi \le x < 2\pi, \end{cases}$$

sketch the sum of its Fourier series for $-2\pi < x < 4\pi$.

[6 marks]

(c) Show that its Fourier series has the following form

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{[1 + (-1)^{n+1}]}{n^2 \pi} \cos(nx) + \frac{(-1)^n}{n} \sin(nx) \right\}.$$

[10 marks]

(d) Hence, by setting $x = \pi$, prove that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

[5 marks]

4. Given that

$$f(x) = \begin{cases} 0, & 0 < x < \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} \leq x < \pi, \end{cases}$$

(a) sketch the sum of its Fourier half range Cosine series for $-3\pi < x < 3\pi$.

[6 marks]

(b) Show that its Fourier half range Cosine series has the following form

$$f(x) = \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \cos(nx).$$

[7 marks]

(c) Sketch the sum of its Fourier half range Sine series for $-3\pi < x < 3\pi$.

[6 marks]

(d) Show that its Fourier half range Sine series has the following form

$$f(x) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right] \sin(nx).$$

[6 marks]

5. (a) Given the Laplace transform of the function $x(t)$ is defined as

$$\mathcal{L}\{x(t)\} = \bar{x}(s) = \int_0^\infty x(t)\, \mathrm{e}^{-st}\, dt,$$

show that

$$\mathcal{L}\{x'(t)\} = s\bar{x}(s) - x(0), \quad \mathcal{L}\{x''(t)\} = s^2\bar{x}(s) - sx(0) - x'(0),$$

where prime ($'$) denotes the derivative, and $x(0)$ and $x'(0)$ denote $x(t)$ and $x'(t)$ at $t = 0$, respectively.

[8 marks]

(b) The quotient

$$\frac{s^2 + 7s + 11}{(s+3)(s+2)^2}$$

can be written as

$$\frac{A}{s+3} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

by the use of partial fractions. Find the constants A, B and C.

[4 marks]

(c) Using (a) and (b) show that the solution of the differential equation

$$\frac{d^2x(t)}{dt^2} + 5\frac{dx(t)}{dt} + 6x(t) = \mathrm{e}^{-2t}, \quad x(0) = 1, \quad \frac{dx}{dt}(0) = 0,$$

is

$$x(t) = -\mathrm{e}^{-3t} + 2\mathrm{e}^{-2t} + t\,\mathrm{e}^{-2t}.$$

Hint: $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$.
First shift theorem: $\mathcal{L}\{x(t)\,\mathrm{e}^{-\beta t}\} = \bar{x}(s+\beta)$.

[13 marks]