THE UNIVERSITY
of LIVERPOOL

Math 295

January 2004 Exam

Engineering Analysis — Year 2 Engineers

Full marks will be awarded for complete answers to FOUR questions. Only the best 4 answers will be taken into account. Note that each question carries a total of 20 marks that are distributed as stated.

```
For this half unit course, the exam accounts for 80%
while homework for 20%.
```

1. **[20 marks]**

(1) A workshop engineer intends to cut a small cylinder of radius r and height h out of a larger steel cylinder of radius R and height H. He knows how to calculate the volume V of the larger cylinder after the cut according to

$$V = V(r, h, R, H) = \pi R^2 H - \pi r^2 h.$$

If the measurements of r, h, R, H are subject to small changes of $\delta r, \delta h, \delta R, \delta H$ respectively, help him to work out a formula for the proportional change δV in the volume V. [10 marks]

(2) Find all the FOUR stationary points of function [5 marks]

$$f = f(x, y) = 2x^3 + 6xy^2 - 3y^2 - 150x,$$

and decide whether one of them (x_o, y_o), nearest to $(5, 1)$, is a local maximum or minimum by checking the signs of $f_{xx}|_{(x_o,y_o)}$ and $(f_{xy}^2 - f_{xx}f_{yy})|_{(x_o,y_o)}$.
[5 marks]

2. **[20 marks]**

(1) For the function $f = f(x, y) = \sin(xy)e^{x-y} - 5$, verify that

$$\begin{aligned} f_{xy} &= e^{x-y}\left[\cos(xy) + x\cos(xy) - y\cos(xy) - \sin(xy) - xy\sin(xy)\right], \\ f_y(\pi, 1) &= -\pi e^{\pi-1} = -\pi\exp(\pi - 1), \quad f_{xx}(\pi, 1) = -2\exp(\pi - 1), \end{aligned}$$

and further find the Taylor expansion (up to and including the second order terms) near the point [10 marks]

$$(a, b) = (\pi, 1).$$

(2) For the surface $z = z(x, y)$ implicitly defined by the following equation

$$\begin{aligned} F(x, y, z) &= 5 - \pi\, e^{\pi-1}\,(y-1) - (x-\pi)\, e^{\pi-1} + \pi\, e^{\pi-1}\,(y-1)^2 \\ &\quad -\pi\, e^{\pi-1}\,(x-\pi)\,(y-1) - (x-\pi)^2\, e^{\pi-1} - z = 0 \end{aligned}$$

near the point

$$p = (x_0, y_0, z_0) = (\pi, 0, 5 + 2\,\pi\, e^{\pi-1}),$$

(2a) compute the tangent plane

$$z - z_0 = (x - x_0) z_x|_p + (y - y_0) z_y|_p$$

that passes through the point p; [7 marks]

(2b) compute the gradient vector ∇F of F at $p = (x_0, y_0, z_0)$. [3 marks]

3. **[20 marks]**

Sketch the function $f(t)$ in $-3\pi < t \leq 3\pi$, when

$$f(t) = \begin{cases} \pi - t, & 0 < t \leq \pi, \\ 0, & -\pi < t \leq 0, \end{cases}$$

is periodic with period $T = 2\pi$. [4 marks]

Further,

(1) State the formulae for all the Fourier coefficients a_0, a_n, b_n for the Fourier series expansion [6 marks]

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right).$$

(2) Verify that the Fourier series for the above function is [10 marks]

$$f(t) = \frac{\pi}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n^2} \cos(nt) + \sum_{n=1}^{\infty} \frac{1}{n} \sin(nt).$$

You may assume $\cos(n\pi) = (-1)^n$.

4. **[20 marks]**

For function $f(x)$ defined by

$$f(x) = \begin{cases} \pi - x, & \pi/2 \le x \le \pi, \\ x, & 0 \le x \le \pi/2, \end{cases}$$

(1) Verify, using the change of variable $y = \pi - x$ and $\cos(n\pi) = (-1)^n$, that

$$\int_{\pi/2}^{\pi} (\pi - x) \sin nx dx = (-1)^{n+1} \int_0^{\pi/2} y \sin ny dy,$$

$$\int_0^{\pi} f(x) \sin(nx) dx = \left[1 + (-1)^{n+1}\right] \int_0^{\pi/2} x \sin nx dx,$$

and further show that [10 marks]

$$\int_0^{\pi} f(x) \sin(nx) dx = \begin{cases} 0, & \text{when } n \ge 0 \text{ is even}, \\ \dfrac{2}{n^2} \sin(\dfrac{n\pi}{2}), & \text{when } n \ge 1 \text{ is odd}, \end{cases}$$

Hint. $\sin(y \pm n\pi) = (-1)^n \sin(y)$ for $n = 1, 2, 3, \ldots$.

(2) Following (1), show that the Fourier half range Since series for $f(x)$ is given by

$$f(x) = \frac{4}{\pi} \left(\sin x - \frac{1}{9} \sin 3x + \frac{1}{25} \sin 5x - \frac{1}{49} \sin 7x + \cdots \right).$$

[6 marks]

(3) Explain how to obtain a Fourier half range Cosine series for $f(x)$ i.e. show how to extend the function and what the proper formulae are (**without** finding the series).

[4 marks]

5. **[20 marks]**

The Laplace transform for function $f(t)$ is defined by

$$\overline{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt.$$

(1) Using the definition, compute [8 marks]

$$\mathcal{L}\{e^{\lambda t} + e^{-\lambda t}\}$$

where λ is some known constant.

(2) Solve the ordinary differential equation for $x = x(t)$ [12 marks]

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = e^{-t}, \qquad x(0) = -1, \quad \frac{dx}{dt}(0) = 1,$$

by using the Laplace transform approach and the properties of the Laplace transform:

$$\begin{cases} \mathcal{L}\{\frac{dx}{dt}(t)\} = s\mathcal{L}\{x(t)\} - x(0), \\ \mathcal{L}\{t^n e^{\alpha t}\} = \dfrac{n!}{(s-\alpha)^{n+1}}, \\ \mathcal{L}\{\frac{d^2x}{dt^2}(t)\} = s^2\mathcal{L}\{x(t)\} - sx(0) - \frac{dx}{dt}(0). \end{cases}$$