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1. (a) Given

$$z(x, y) = \frac{x^2 y^3}{x^5 + y^5},$$

show that

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0.$$

[6 marks]

(b) For the function $f(x, y) = x^3 e^{(2x-y)}$, find the Taylor polynomial of second order $p_2(x, y)$ near $(1, 1)$, verifying first that

$$\begin{aligned} f_{xy}(x, y) &= -3x^2 e^{(2x-y)} - 2x^3 e^{(2x-y)}, \\ f_x(1, 1) &= 5e, \quad f_{yy}(1, 1) = e. \end{aligned}$$

[14 marks]

2. (a) State the conditions for a point (x_0, y_0) to be a saddle point of a function $f(x, y)$. [4 marks]

(b) Find all the stationary points of the function

$$f(x, y) = 4x^3 - 4xy + 2y^2,$$

and for each of them determine whether it is a minimum, a maximum or a saddle point. [8 marks]

(c) Consider the implicit function $z = z(x, y)$ determined by

$$F(x, y, z) = \cos(z) - z^3 + x^2 y - 3y^2 = 0.$$

Use implicit differentiation to evaluate z_x and z_y at the point $P_0 = (1, \sqrt{4 + \pi^3}, \pi)$ and hence find the equation of the tangent plane to the surface $z = z(x, y)$ at P_0 . [8 marks]



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3. (a) State the formula for the Fourier series of the function

$$f(x), \quad 0 \leq x \leq 2\pi,$$

of period 2π and the formulae to determine the Fourier coefficients

$$a_0, a_n, b_n, \quad n = 1, 2, \dots$$

[6 marks]

- (b) Given the periodic function $f(x)$ with period $T = 2\pi$

$$f(x) = \begin{cases} x - \pi, & 0 \leq x < \pi, \\ 0, & \pi \leq x < 2\pi, \end{cases}$$

sketch the sum of its Fourier series for $-2\pi < x < 2\pi$.

[4 marks]

- (c) Verify that the Fourier series for the above function is

$$f(x) = -\frac{\pi}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n^2} [(-1)^n - 1] \cos(nx) - \frac{1}{n} \sin(nx) \right)$$

Hint: You may assume that $\cos(n\pi) = (-1)^n$.

[10 marks]



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4. Let $f(x)$ be the function

$$f(x) = \begin{cases} 1, & 0 \leq x < \pi/2, \\ \pi, & \pi/2 \leq x \leq \pi. \end{cases}$$

(a) Sketch the graph of the function $f(x)$ and by extending it to an even function in the interval $[-\pi, \pi]$, sketch the graph of the sum of its Fourier half range Cosine series in the interval $[-2\pi, 2\pi]$. [6 marks]

(b) Verify that the Fourier half range Cosine series has the form

$$f(x) = \frac{1 + \pi}{2} + \frac{2(1 - \pi)}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\pi/2) \cos(nx).$$

[7 marks]

(c) Using (b) and setting $x = 0$, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin(n\pi/2) = \frac{\pi}{4}.$$

[3 marks]

(d) Sketch the graph of the sum of the Fourier half range Sine series for $f(x)$ in the interval $[-\pi, \pi]$ WITHOUT calculating the Fourier coefficients.

[4 marks]



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5. Given the definition of the Laplace Transform for a function $f(t)$,

$$\overline{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt, \quad s > 0,$$

- (a) Prove that $\mathcal{L}\{\cos(t)\} = \frac{s}{s^2 + 1}$ and $\mathcal{L}\{\sin(t)\} = \frac{1}{s^2 + 1}$.

Hint: $e^{it} = \cos(t) + i\sin(t)$ and the Laplace transform is linear, i.e,

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}, \quad a, b \text{ real numbers.}$$

[10 marks]

- (b) Solve the following ordinary differential equation for $x(t)$,

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 5x(t) = 3e^{4t}, \quad x(0) = 1, \quad \frac{dx}{dt}(0) = 0.$$

Hint: Verify that

$$\overline{x}(s) = \frac{s^2 - 10s + 27}{(s - 4)(s^2 - 6s + 5)}$$

by using the following properties of the Laplace transform

$$\begin{cases} \mathcal{L}\{e^{\alpha t}\} = \frac{1}{s - \alpha} \\ \mathcal{L}\left\{\frac{dx}{dt}(t)\right\} = s\mathcal{L}\{x(t)\} - x(0), \\ \mathcal{L}\left\{\frac{d^2x}{dt^2}(t)\right\} = s^2\mathcal{L}\{x(t)\} - sx(0) - \frac{dx}{dt}(0). \end{cases}$$

[10 marks]