PAPER CODE NO.
MATH293

THE UNIVERSITY
of LIVERPOOL

JANUARY 2007 EXAMINATIONS

Bachelor of Engineering : Year 2
Master of Engineering : Year 2

ENGINEERING MATHEMATICS I

TIME ALLOWED : Two Hours and a half

INSTRUCTIONS TO CANDIDATES:
Full marks can be obtained for complete answers to FOUR questions.
Only the best FOUR answers will be counted.

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1. (a) Find without using the Laplace transform, the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0.$$

[6 marks]

(b) Find a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^x \ .$$

Hence solve the equation with the initial conditions

$$y(0) = 0, \quad y'(0) = 1 \ .$$

[13 marks]

(c) Suggest the form of the trial solution in the method of undetermined coefficients, for the equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = \sin 2x + x \sin 4x \ .$$

(You are not required to do the calculations here, i.e. you should leave the coefficients in the trial solution undetermined.)

[6 marks]

2. (a) The function $f(x)$ is periodic, with period $P = \pi$, and has the Fourier series expansion

$$f(x) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos(2nx) + b_n \sin(2nx)\} .$$

State the formulae for the Fourier coefficients a_0, a_n, and b_n, where $n = 1, 2, \ldots$, valid for this period.

[7 marks]

(b) The function $g(x)$ is defined by

$$g(x) = \begin{cases} 1, & -\pi/2 < x < 0, \\ -1, & 0 < x < \pi/2, \end{cases}$$

$$\text{with} \qquad g(x) = g(x + \pi), \qquad \text{for all } x.$$

Sketch the graph of $g(x)$ for $-2\pi < x < 2\pi$.

[3 marks]

(c) Find the Fourier series of the function $g(x)$ defined in part (b). Write out explicitly the partial sum of the series up to and including terms with $\cos(10x)$ and/or $\sin(10x)$.

[12 marks]

(d) Calculate the value of this partial sum at $x = \pi/4$ to four decimal places and estimate the relative accuracy with which it approximates the exact value of $g(\pi/4)$.

[3 marks]

3. (a) The Fourier transform $\tilde{f}(\omega)$ of a function $f(t)$ is defined by the formula

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \ .$$

Give the formula by which the function $f(t)$ can be found from its Fourier transform $\tilde{f}(\omega)$.

[6 marks]

(b) A function $g(t)$ is given by

$$g(t) = \begin{cases} 0, & t < -b, \\ 3, & -b \leq t \leq b, \\ 0, & b < t, \end{cases}$$

where b is a positive constant. Show that its Fourier transform is

$$\tilde{g}(\omega) = \frac{6}{\sqrt{2\pi}} \frac{\sin(b\omega)}{\omega} \ .$$

Sketch the functions $g(t)$ and $\tilde{g}(\omega)$, given that $\frac{\sin(b\omega)}{\omega}$ tends to the value b when $\omega = 0$.

[15 marks]

(c) Using the results of parts (a) and (b), write down the integral which represents $g(t)$ defined in part (b).

[4 marks]

4. (a) Find the function of t whose Laplace transform is:

$$\frac{2}{s(s+5)}.$$

[4 marks]

(b) Find the function of t whose Laplace transform is:

$$\frac{s+1}{s^2-4s+8}.$$

[8 marks]

(c) Find, **using the Laplace Transform**, the solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+2\frac{dy}{dt}+y=t,$$

which satisfies initial conditions

$$y(0)=0 \quad \text{and} \quad y'(0)=0.$$

Check your solution by substituting $y(t)$ into the differential equation and initial conditions.

[13 marks]

Note: a table of standard Laplace transforms is available on page 7.

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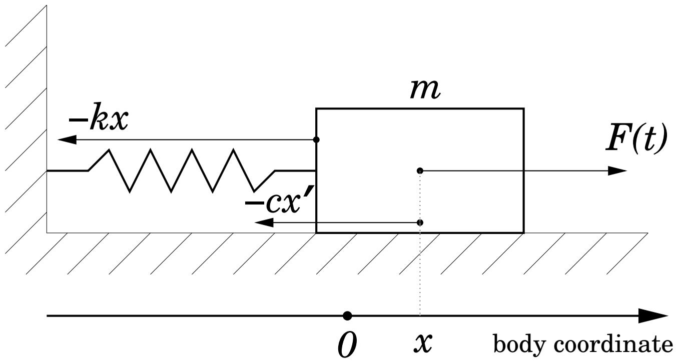
5. A body of mass $m = 2\,\mathrm{kg}$ is attached to a spring and can move along a horizontal surface (see the diagram). The horizontal coordinate x of the body is measured with respect to its equilibrium position. The spring has the Hooke's spring constant of $k = 1\,\mathrm{N\,m^{-1}}$. The frictional force acting on the body is proportional to its velocity with the coefficient $c = 3\,\mathrm{N\,s\,m^{-1}}$. In addition, the body is affected by a periodic external force, changing with time according to the law $F(t) = F_0 \sin(\omega t)$, where $F_0 = 30\,\mathrm{N}$ and $\omega = 1\,\mathrm{s^{-1}}$.



(a) Write down a differential equation for $x(t)$, the position of the body.
[5 marks]

(b) Using the Laplace transform, or otherwise, find the solution to this equation for the initial conditions $x(0) = 0$, $x'(0) = 0$.
[16 marks]

Note: a table of standard Laplace transforms is available on page 7.

(c) Represent this solution as a sum of free movement and forced oscillations. Based on the form of free movement, or otherwise, classify this system as underdamped, critically damped or overdamped.
[4 marks]

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Table of Laplace transforms

$f(t)$ (original)	$F(s)$ (image)	$f(t)$ (original)	$F(s)$ (image)
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
t	$\frac{1}{s^2}$	te^{at}	$\frac{1}{(s-a)^2}$
t^2	$\frac{2}{s^3}$	t^2e^{at}	$\frac{2}{(s-a)^3}$
t^n	$\frac{n!}{s^{n+1}}$	t^ne^{at}	$\frac{n!}{(s-a)^{n+1}}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$\frac{t\sin(\omega t)}{2\omega}$	$\frac{s}{(s^2+\omega^2)^2}$	$\frac{t\sin(\omega t)}{2\omega}e^{at}$	$\frac{s-a}{((s-a)^2+\omega^2)^2}$
$\frac{\sin(\omega t)-\omega t\cos(\omega t)}{2\omega^2}$	$\frac{\omega}{(s^2+\omega^2)^2}$	$\frac{\sin(\omega t)-\omega t\cos(\omega t)}{2\omega^2}e^{at}$	$\frac{\omega}{((s-a)^2+\omega^2)^2}$
$y(t)$	$Y(s)$	$e^{at}y(t)$	$Y(s-a)$
$\frac{\mathrm{d}y(t)}{\mathrm{d}t}$	$sY(s)-y(0)$	$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}$	$s^2Y(s)-sy(0)-y'(0)$