PAPER CODE NO.
MATH293

THE UNIVERSITY
of LIVERPOOL

JANUARY 2006 EXAMINATIONS

Bachelor of Engineering : Year 2
Master of Engineering : Year 2

ENGINEERING MATHEMATICS I

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES:
Full marks can be obtained for complete answers to FOUR questions.
Only the best FOUR answers will be counted.

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1. (a) Find without using the Laplace transform, the general solution of the differential equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0.$$

[**7 marks**]

(b) Using the method of undetermined coefficients, find a particular integral of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 6x^2 + 10x - 10\ .$$

Hence write down the general solution of the equation.

[**12 marks**]

(c) Suggest the form of the trial solution in the method of undetermined coefficients, for the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = (x^2 + x)e^{-x}\ .$$

(You are not required to do the calculations here, i.e. you should leave the coefficients in the trial solution undetermined.)

[**6 marks**]

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2. (a) The function $f(x)$ is periodic, with period $P = 2$, and has the Fourier series expansion

$$f(x) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos(n\pi x) + b_n \sin(n\pi x)\} .$$

State the formulae for the Fourier coefficients a_0, a_n, and b_n, $n = 1, 2, \ldots$, valid for this period.

[7 marks]

(b) The function $f(x)$ is defined by

$$f(x) = \begin{cases} -x, & -1 \leq x \leq 0, \\ x, & 0 \leq x \leq 1, \end{cases}$$

$$\text{with} \qquad f(x) = f(x+2), \qquad \text{for all } x.$$

Sketch the graph of $f(x)$ for $-4 < x < 4$. Explain why the function $f(x)$ defined above is even.

[5 marks]

(c) Find the Fourier series of the function $f(x)$ defined in part (b). You may use the result

$$\int x \cos(ax)\, dx = \frac{x}{a} \sin(ax) + \frac{1}{a^2} \cos(ax) + C$$

where a is a non-zero constant and C is a constant of integration. Write out explicitly the partial sum of the series up to and including terms with $\cos(5\pi x)$ and/or $\sin(5\pi x)$.

[10 marks]

(d) Calculate the value of this partial sum at $x = 1/4$ to four decimal places and estimate the relative accuracy with which it approximates the exact value of $f(1/4)$.

[3 marks]

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3. (a) The Fourier transform $\tilde{f}(\omega)$ of a function $f(t)$ is defined by the formula

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \ .$$

Give the formula by which the function $f(t)$ can be found from its Fourier transform $\tilde{f}(\omega)$.

[6 marks]

(b) Show that if $f(t)$ is given by

$$f(t) = \begin{cases} 3\,e^{-2t}, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

its Fourier transform is

$$\frac{3}{\sqrt{2\pi}} \frac{1}{2 + i\omega}$$

[15 marks]

(c) Using the results of parts (a) and (b), write down the integral which represents $f(t)$ defined in part (b).

[4 marks]

4. (a) Find the function of t whose Laplace transform is:

$$\frac{1}{(s+2)(s+3)}.$$

[4 marks]

(b) Find the function of t whose Laplace transform is:

$$\frac{2s+1}{s^2+4s+8}.$$

[8 marks]

(c) Find, **using the Laplace Transform**, the solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 4y = 9e^{-t},$$

which satisfies initial conditions

$$y(0) = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t}(0) = 3.$$

Check your solution by substituting $y(t)$ into the differential equation and initial conditions.

[13 marks]

Note: a table of standard Laplace transforms is available on page 7.

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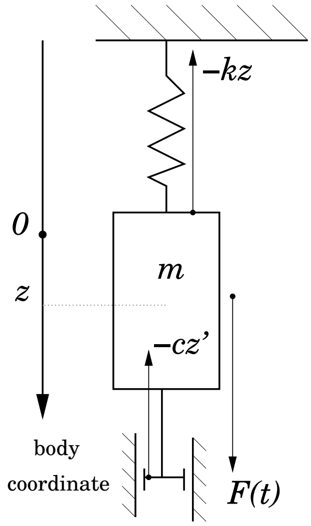
5. A body of mass $m = 1\,\text{kg}$ is suspended on a spring and attached to a dash-pot (see the diagram). The vertical coordinate z of the body is measured with respect to its equilibrium position. The spring has the Hooke's spring constant of $k = 2\,\text{N}\,\text{m}^{-1}$. The frictional force acting on the body is proportional to its velocity with the coefficient $c = 3\,\text{N}\,\text{s}\,\text{m}^{-1}$. In addition, the body is affected by a periodic external force, changing with time according to the law $F(t) = F_0 \sin(\omega t)$ where $F_0 = 20\,\text{N}$ and $\omega = 2\,\text{s}^{-1}$.



(a) Write down a differential equation for the vertical coordinate $z(t)$ of the body.

[5 marks]

(b) Using the Laplace transform, or otherwise, find the solution to this equation for the initial conditions $z(0) = 0$, $\mathrm{d}z/\mathrm{d}t(0) = 0$.

[17 marks]

Note: a table of standard Laplace transforms is available on page 7.

(c) Represent this solution as a sum of free movement and forced oscillations. Based on the form of free movement, or otherwise, classify this system as underdamped, critically damped or overdamped.

[3 marks]

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Table of Laplace transforms

$f(t)$ (original)	$F(s)$ (image)	$f(t)$ (original)	$F(s)$ (image)
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
t	$\frac{1}{s^2}$	te^{at}	$\frac{1}{(s-a)^2}$
t^2	$\frac{2}{s^3}$	t^2e^{at}	$\frac{2}{(s-a)^3}$
t^n	$\frac{n!}{s^{n+1}}$	t^ne^{at}	$\frac{n!}{(s-a)^{n+1}}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$\frac{t\sin(\omega t)}{2\omega}$	$\frac{s}{(s^2+\omega^2)^2}$	$\frac{t\sin(\omega t)}{2\omega}e^{at}$	$\frac{s-a}{((s-a)^2+\omega^2)^2}$
$\frac{\sin(\omega t)-\omega t\cos(\omega t)}{2\omega^2}$	$\frac{\omega}{(s^2+\omega^2)^2}$	$\frac{\sin(\omega t)-\omega t\cos(\omega t)}{2\omega^2}e^{at}$	$\frac{\omega}{((s-a)^2+\omega^2)^2}$
$y(t)$	$Y(s)$	$e^{at}y(t)$	$Y(s-a)$
$\frac{\mathrm{d}y(t)}{\mathrm{d}t}$	$sY(s)-y(0)$	$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}$	$s^2Y(s)-sy(0)-y'(0)$