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THE UNIVERSITY
of LIVERPOOL

JANUARY 2005 EXAMINATIONS

Bachelor of Engineering : Year 2
Master of Engineering : Year 2

ENGINEERING MATHEMATICS I

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FOUR questions.
Only the best FOUR answers will be counted.

1. (a) Find, without using the Laplace transform, the general solution of the differential equation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0.$$

(7 marks)

(b) Using the method of undetermined coefficients, find a particular integral of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 3e^{-x}.$$

Hence write down the general solution of the equation.

(12 marks)

(c) Suggest the form of the trial solution in the method of undetermined coefficients, for the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 5e^{x} + 6\sin(2x).$$

(You are not required to do the calculations here, i.e. you should leave the coefficients in the trial solution undetermined.)

(6 marks)

2. (a) The function $f(x)$ is periodic, with period $p = 2L = 2\pi$, and has the Fourier series expansion

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

State the formulae for the Fourier coefficients, a_0, a_n, and b_n, $n = 1, 2, \ldots$, valid for this period.

(7 marks)

(b) The function $f(x)$ is defined by

$$f(x) = \begin{cases} -\pi - x, & -\pi < x < 0, \\ \pi - x, & 0 < x < \pi, \\ f(x \pm 2\pi), & \text{for all } x. \end{cases}$$

Sketch the graph of $f(x)$ for $-3\pi < x < 3\pi$. Give the definition of an odd function. Explain what special features a Fourier series of an odd function has. Explain why the function $f(x)$ defined above is odd.

(6 marks)

(c) Find the Fourier series of the function $f(x)$ defined in part (b). You may use the following result:

$$\int (\pi - x) \sin(kx) \, \mathrm{d}x = -\frac{\pi - x}{k} \cos(kx) - \frac{1}{k^2} \sin(kx)$$

where k is a non-zero constant. Write out explicitly the partial sum of this series up to and including terms with $\cos(7x)$ and/or $\sin(7x)$.

(9 marks)

Calculate the value of this partial sum at $x = \pi/2$ to four decimal places and estimate the relative accuracy with which it approximates the exact value of $f(\pi/2)$.

(3 marks)

3. (a) A given function $f(t)$ can be represented using its Fourier transform $\tilde{f}(w)$ by the formula

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(w) e^{iwt} \, \mathrm{d}w.$$

Give the formula, by which the Fourier transform $\tilde{f}(w)$ can be found if the function $f(t)$ is known.

(6 marks)

(b) Show that if $f(t)$ is given by

$$f(t) = \begin{cases} t/2, & 0 < t < 2, \\ 0, & \text{otherwise} \end{cases},$$

its Fourier transform is

$$\tilde{f}(w) = \frac{1}{2w^2\sqrt{2\pi}} \left((2iw + 1)e^{-2iw} - 1 \right)$$

To do that, you may use the following result:

$$\int te^{kt} \, \mathrm{d}t = \left(\frac{t}{k} - \frac{1}{k^2} \right) e^{kt},$$

where k is a non-zero constant.

(15 marks)

(c) Using the results of parts (a) and (b), write down the integral which represents $f(t)$ defined in part (b).

(4 marks)

4. (a) Find the function of t whose Laplace transform is:

$$\frac{1}{(s+1)(s+4)}$$

(4 marks)

(b) Find the function of t whose Laplace transform is:

$$\frac{s+7}{s^2+6s+13}$$

(8 marks)

(c) Find, **using the Laplace Transform**, the solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+9y=16\cos(5t),$$

which satisfies initial conditions

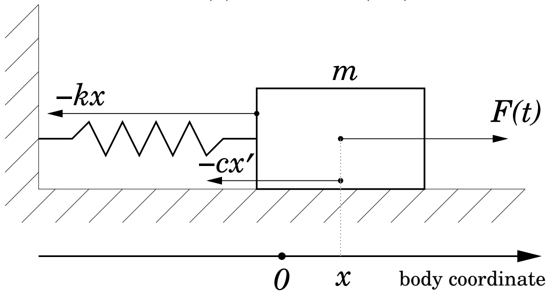
$$y(0)=0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t}(0)=3.$$

Check your solution by substituting $y(t)$ into the differential equation and initial conditions.

(13 marks)

Note: a table of standard Laplace transforms is available on page 7.

5. A body of mass $m = 1\,\mathrm{kg}$ is placed on a horizontal surface and attached to a spring with the Hooke's spring constant of $k = 1\,\mathrm{N/m}$ (see the diagram). The horizontal coordinate x of the body is measured with respect to the equilibrium position of the spring. The frictional force acting on the body is proportional to its velocity with the coefficient $c = 2\,\mathrm{Ns/m}$. The external horizontal force applied to the body is changing with time according to the law $F(t) = F_0 \cos(\omega t)$ where $F_0 = 50\,\mathrm{N}$ and $\omega = 3\,\mathrm{s}^{-1}$.



(a) Write down a differential equation for the coordinate $x(t)$ of the body.

(5 marks)

(b) Using the Laplace transform, or otherwise, find the solution to this equation for the initial conditions $x(0) = 0$, $\mathrm{d}x/\mathrm{d}t(0) = 0$.

(15 marks)

Note: a table of standard Laplace transforms is available on page 7.

(c) Represent this solution as a sum of the free movement and the forced oscillations. Based on the form of the free movement, or otherwise, classify this system as underdamped, critically damped or overdamped. Find the coordinate x of the body at the time $t = 10\pi\,\mathrm{s}$, with the precision of 5 significant figures.

(5 marks)

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Table of Laplace transforms

$f(t)$ (original)	$F(s)$ (image)	$f(t)$ (original)	$F(s)$ (image)
1	$\dfrac{1}{s}$	e^{at}	$\dfrac{1}{s-a}$
t	$\dfrac{1}{s^2}$	te^{at}	$\dfrac{1}{(s-a)^2}$
t^2	$\dfrac{2}{s^3}$	t^2e^{at}	$\dfrac{2}{(s-a)^3}$
t^n	$\dfrac{n!}{s^{n+1}}$	t^ne^{at}	$\dfrac{n!}{(s-a)^{n+1}}$
$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$	$e^{at}\cos(\omega t)$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$	$e^{at}\sin(\omega t)$	$\dfrac{\omega}{(s-a)^2+\omega^2}$
$\dfrac{t\sin(\omega t)}{2\omega}$	$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t\sin(\omega t)}{2\omega}e^{at}$	$\dfrac{s-a}{((s-a)^2+\omega^2)^2}$
$\dfrac{\sin(\omega t)-\omega t\cos(\omega t)}{2\omega^2}$	$\dfrac{\omega}{(s^2+\omega^2)^2}$	$\dfrac{\sin(\omega t)-\omega t\cos(\omega t)}{2\omega^2}e^{at}$	$\dfrac{\omega}{((s-a)^2+\omega^2)^2}$
$y(t)$	$Y(s)$	$e^{at}y(t)$	$Y(s-a)$
$\dfrac{\mathrm{d}y(t)}{\mathrm{d}t}$	$sY(s)-y(0)$	$\dfrac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}$	$s^2Y(s)-sy(0)-y'(0)$