Math293 January 2002 exam: solutions

1. (a) **Question** *Find, without using the Laplace transform, the general solution of the differential equation:*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0.$$

Answer The characteristic equation is

$$\lambda^2 + 4\lambda + 5 = 0,$$

and its roots are

$$\lambda_{1,2} = -2 \pm i,$$

(negative discriminant, two complex conjugate roots). Hence the general solution of homogeneous equation is

$$y = \boxed{(C_1 \cos x + C_2 \sin x)\, e^{-2x}}.$$

7 marks for this part

(b) **Question** *Using the method of undetermined coefficients, find a particular integral of the differential equation*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 5x^2 + 3x + 3.$$

Hence write down the general solution of the equation.

Answer The free term here is a quadratic polynomial, the trial solution to be chosen as a quadratic polynomial,

$$y_p = Ax^2 + Bx + C.$$

Then

$$y_p' = 2Ax + B,$$
$$y_p'' = 2A$$

Substitution into the equation gives

$$\begin{aligned} \text{LHS} = y_p'' + 4y_p'' + 5y_p &= 2A + 4(2Ax + B) + 5(Ax^2 + Bx + C) \\ &= 5Ax^2 + (8A + 5B)x + (2A + 4B + 5C) \\ &= \text{RHS} = 5x^2 + 3x + 3 \end{aligned}$$

Equating the coefficients at the same powers of x leads to

$$\begin{aligned} [x^2]: &\quad 5A = 5, \\ [x]: &\quad 8A + 5B = 3, \\ [\text{const}]: &\quad 2A + 4B + 5C = 3 \end{aligned}$$

The solution of this system is

$$A = 1, \quad B = -1, \quad C = 1$$

and so the particular solution to the nonhomogeneous equation is

$$y_p = x^2 - x + 1.$$

The general solution of the nonhomogeneous equation is the sum of the particular solution and the complementary function (GSHE), that is,

$$y = \boxed{(C_1 \cos x + C_2 \sin x)\, e^{-2x} + x^2 - x + 1}.$$

13 marks for this part

(c) **Question** *Suggest the form of the trial solution in the method of undetermined coefficients, for the equation*

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = (5x^2 + 3x + 3)\sin(2x).$$

(You don't have to do the calculations here, i.e. you should leave the coefficients in the trial solution undetermined.)

Answer

$$y_p = \boxed{(Ax^2 + Bx + C)\cos(2x) + (Dx^2 + Ex + F)\sin(2x)}.$$

5 marks for this part

Total for this question: 25 marks

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2. (a) **Question** *The function $f(x)$ is periodic, with period $p = 2L = 2$, and has the Fourier series expansion*

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(n\pi x) + b_n\sin(n\pi x)).$$

State the formulae for the Fourier coefficients, a_0, a_n, $n = 1, 2, \dots$ and b_n, $n = 1, 2, \dots$, valid for this period.

Answer

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx,$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx, \qquad n = 1, 2 \dots,$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx, \qquad n = 1, 2 \dots$$

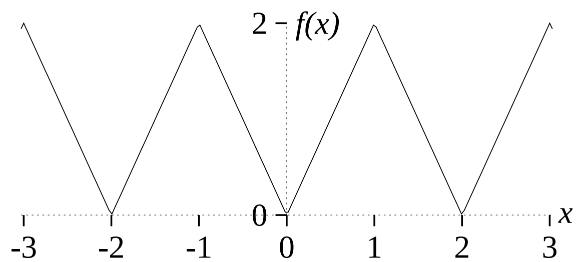
7 marks for this part

(b) **Question** *The function $f(x)$ is defined by*

$$f(x) = \begin{cases} -2x, & -1 \le x \le 0, \\ 2x, & 0 \le x \le 1, \\ f(x \pm 2), & \textit{for all } x. \end{cases}$$

Sketch the graph of $f(x)$ for $-3 < x < 3$.

Answer



Question *Give the definition of an even function.*

Answer $f(-x) = f(x)$ for all x.

Question *Explain what special features a Fourier series of an even function has.*

Answer It lacks all sin terms.

Question *Explain why the function $f(x)$ defined above is even.*

Answer Graphical: the graph is mirror-symmetric about the vertical axis.
Analytical: it is even by definition within the symmetric interval $-1 \le x \le 1$, and periodic with period 2 equal to the length of that interval, therefore even everywhere.

6 marks for this part

(c) **Question** *Find the Fourier series of the function $f(x)$ defined above. You may use the following result:* $\int x\cos(kx)\,\mathrm{d}x = \frac{x}{k}\sin(kx) + \frac{1}{k^2}\cos(kx)$, *where $k \neq 0$ is any constant.*

Answer

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2} * 2\int_{0}^{1} 2x\,\mathrm{d}x$$

$$= 2\left(\frac{x^2}{2}\right)_0^1 = 1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \int_{-1}^{1} f(x)\cos(n\pi x)\,\mathrm{d}x$$

$$= 2\int_{0}^{1} f(x)\cos(n\pi x)\,\mathrm{d}x = 2\int_{0}^{1} 2x\cos(n\pi x)\,\mathrm{d}x$$

$$= 4\left(\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{n^2\pi^2}\cos(n\pi x)\right)_0^1 = 4\,\frac{1}{n^2\pi^2}\,(\cos(n\pi) - 1)$$

$$= -\frac{4}{n^2\pi^2}\,(1 - (-1)^n) = \begin{cases} \frac{8}{n^2\pi^2}, & \text{for odd } n, \\ 0, & \text{for even } n, \end{cases}$$

Thus,

$$f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1,3,5\ldots} \frac{1}{n^2}\cos(n\pi x)$$

Question *Write out explicitly the partial sum of this series up to terms with $\cos(5\pi x)$ and/or $\sin(5\pi x)$.*

Answer

$$f(x) \approx S_5(x) = 1 - \frac{8}{\pi^2}\left(\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(5\pi x)\right)$$

9 marks for this part

Question *Calculate the value of this partial sum at $x = 1$ at 3 d.p. and estimate the relative accuracy with which it approximates the the exact value of $f(x)$ at that point.*

Answer

$$S_5(1) = 1 - \frac{8}{\pi^2}\left(\cos(\pi) + \frac{1}{9}\cos(3\pi) + \frac{1}{25}\cos(5\pi)\right) = 1 + \frac{8}{\pi^2}\left(1 + \frac{1}{9} + \frac{1}{25}\right) \approx 1.933$$

$$f(1) = 2$$

thus the relative error is $\left|\frac{f(1)-S_5(1)}{f(1)}\right| \approx \frac{0.067}{2} \approx \boxed{3.3\%}$.

3 marks for this part

Total for this question: 25 marks

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3. (a) **Question** *The Fourier transform $\tilde{f}(w)$ of a function $f(t)$ is defined by*

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-iwt}\,\mathrm{d}t.$$

Give the formula, by which $f(t)$ can be found if its Fourier transform $\tilde{f}(w)$ is known.

Answer

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(w)e^{iwt}\,\mathrm{d}w.$$

General structure , correct coefficient , correct limits , correct sign in the exponent .

6 marks for this part

(b) **Question** *Show that if $f(t)$ is given by*

$$f(t) = \begin{cases} 1, & 0 < t < 2, \\ 0, & \textit{otherwise} \end{cases},$$

its Fourier transform is

$$\tilde{f}(w) = \frac{i}{w\sqrt{2\pi}}\left(e^{-2iw} - 1\right)$$

Answer For the given $f(t)$, the Fourier transform is

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-iwt}\,\mathrm{d}t$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{2} e^{-iwt}\,\mathrm{d}t$$

$$= \frac{1}{\sqrt{2\pi}} \frac{i}{w} e^{-iwt}\Big|_0^2$$

$$= \frac{i}{w\sqrt{2\pi}}\left(e^{-2iw} - 1\right)$$

as requested.

13 marks for this part

(c) **Question** *Using the result of parts (a) and (b), write down the integral which represents $f(t)$ defined as above.*

Answer Inverse F.t.:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(w)e^{iwt}\,\mathrm{d}w = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{w}\left(e^{-2iw} - 1\right)e^{iwt}\,\mathrm{d}w$$

Question

By putting $t = 1$ in this result, evaluate

$$\int_0^\infty \frac{\sin w}{w}\, \mathrm{d}w.$$

Answer For $t = 1$,

$$f(1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{w} \left(e^{-2iw} - 1\right) e^{iw}\, \mathrm{d}w$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{w} \left(e^{-iw} - e^{iw}\right) \mathrm{d}w$$

$$= \frac{1}{\pi} \int_0^\infty \frac{i}{w} \left(e^{-iw} - e^{iw}\right) \mathrm{d}w = \frac{1}{\pi} \int_0^\infty \frac{2\sin(w)}{w}\, \mathrm{d}w = f(1) = 1$$

hence

$$\int_0^\infty \frac{\sin(w)}{w}\, \mathrm{d}w = \frac{\pi}{2}$$

6 marks for this part

Total for this question: 25 marks

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4. (a) **Question** *Find the function of t whose Laplace transform is:*

$$\frac{1}{(s-3)(s+2)}$$

Answer By cover-up rule,

$$F(s) = \frac{1}{(s-3)(s+2)} = \frac{1}{(-2)-3} \cdot \frac{1}{s+2} + \frac{1}{3+2} \cdot \frac{1}{s-3}$$

$$= \frac{1}{5}\frac{1}{s-3} - \frac{1}{5}\frac{1}{s+2}$$

(any other valid method equally acceptable)

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \boxed{\frac{1}{5}\left(e^{3t} - e^{-2t}\right)}$$

4 marks for this part

(b) **Question** *Find the function of t whose Laplace transform is:*

$$\frac{s-1}{s^2+2s+5}$$

Answer

$$F(s) = \frac{s-1}{s^2+2s+5} = \frac{s-1}{s^2+2s+1+4} = \frac{(s+1)-2}{(s+1)^2+2^2}$$

$$= \frac{(s+1)}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2}$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{(s+1)}{(s+1)^2+2^2}\right] - \mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+2^2}\right] = e^{-t}\cos 2t - e^{-t}\sin 2t$$

8 marks for this part

(c) **Question** *Find, **using the Laplace Transform**, the solution of the differential equation*

$$\frac{d^2y}{dt^2} + 4y = \cos(2t),$$

which satisfies initial conditions

$$y(0) = 0 \quad \textit{and} \quad \frac{dy}{dt}(0) = 1.$$

Answer Let $\mathcal{L}[y] = Y$, then with account of the initial conditions,

$$\mathcal{L}[y]'' = s^2Y - sy(0) - y'(0) =$$
$$s^2Y - 1,$$

and the subsidiary equation is

$$s^2Y - 1 + 4Y = \frac{s}{s^2+2^2}$$

Its solution is:

$$Y = \frac{s}{(s^2+4)^2} + \frac{1}{s^2+4}$$
$$= \frac{s}{(s^2+2^2)^2} + \frac{1}{2}\frac{2}{s^2+2^2}$$

Thus

$$y = \mathcal{L}^{-1}[Y] = \frac{t\sin(2t)}{2*2} + \frac{1}{2}\sin(2t) = \boxed{\frac{1}{4}(2+t)\sin(2t)}$$

13 marks for this part

Total for this question: 25 marks

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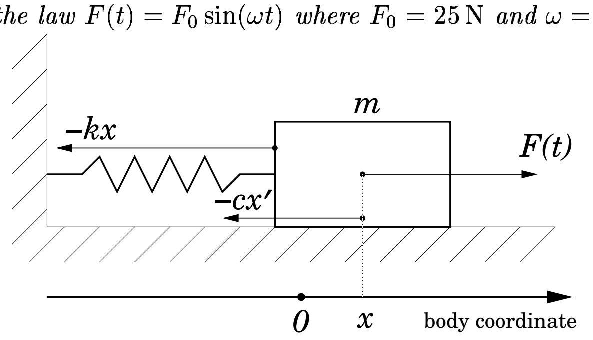
5. *A body of mass $m = 1\,\text{kg}$ is placed on a horizontal surface and attached to a spring with the Hooke's spring constant of $k = 1\,\text{N/m}$ (see the diagram). The horizontal coordinate x of the body is measured with respect to the equilibrium position of the spring. The frictional force acting on the body is proportional to its velocity with the coefficient $c = 2\,\text{Ns/m}$. The external horizontal force applied to the body is changing with time according to the law $F(t) = F_0\sin(\omega t)$ where $F_0 = 25\,\text{N}$ and $\omega = 2\,\text{s}^{-1}$.*



(a) **Question** *Write down a differential equation for the coordinate $x(t)$ of the body.*

Answer

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 25\sin(2t)$$

5 marks for this part

(b) **Question** *Using the Laplace transform, or otherwise, find the solution to this equation for the initial conditions* $x(0) = 0$, $\mathrm{d}x/\mathrm{d}t(0) = 0$.

Answer variant 1. The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$, which has a double root $\lambda_= - 1$, and the complementary functions is

$$x_h = (A + Bt)e^{-t} \qquad .$$

The trial solution $x_p = C\cos(2t) + D\sin(2t)$. Thus $x_p' = 2D\cos(2t) - 2C\sin(2t)$, $x_p'' = -4C\cos(2t) - 4D\sin(2t)$, substitution into the equation gives

$$(-4C + 4D + C)\cos(2t) + (-4D - 4C + D)\sin(2t) = 25\sin(2t)$$

which leads to the system

$$\begin{aligned} -3C + 4D &= 0 \\ -4C - 3D &= 25 \end{aligned}$$

the solution of which is $C = -4$, $D = -3$. Thus the general solution of the equation is

$$x = x_p + x_h = (A + Bt)e^{-t} - 4\cos(2t) - 3\sin(2t)$$

The corresponding velocity is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (B - A - Bt)e^{-t} - 6\cos(2t) + 8\sin(2t)$$

Substituting this into the initial conditions, we get

$$\begin{aligned} A - 4 &= 0 \\ B - A - 6 &= 0 \end{aligned}$$

the solution of which is $A = 4$, $B = 10$, and the ultimate answer is

$$x(t) = (4 + 10t)e^{-t} - 4\cos(t) - 3\sin(2t)$$

Answer variant 2. Let $\mathcal{L}[x(t)] = X(s)$. Then $\mathcal{L}[x'(t)] = sX(s) - x(0) = sX$, $\mathcal{L}[x''(t)] = s^2X - sx(0) - x'(0) = s^2X$, $\mathcal{L}[25\sin(2t)] = 25\frac{2}{s^2+2^2} = \frac{50}{s^2+4}$. Then the Laplace transform of the equation is

$$s^2X + 2sX + X = \frac{50}{s^2 + 2^2}$$

The solution of this equation is

$$X(s) = \frac{50}{(s^2 + 2^2)(s + 1)^2}$$

To find the inverse Laplace transform of $X(s)$, it ought to be brought to the sum of partial fractions,

$$X(s) = \frac{As + B}{s^2 + 4} + \frac{C(s + 1) + D}{(s + 1)^2},$$

bringing this to a common denominator produces

$$\begin{aligned} X(s) = \frac{(A + C)s^3 + (2A + B + C + D)s^2 + (A + 2B + 4C)s + (B + 4C + 4D)}{(s^2 + 4)(s + 1)^2} \\ = \frac{50}{(s^2 + 4)(s + 1)^2} \end{aligned}$$

Equating the coefficients at powers of s in the numerators, this gives the system

$$\begin{aligned} A + C &= 0 \\ 2A + B + c + D &= 0 \\ A + 2B + 4C &= 0 \\ B = 4C + 4D &= 50 \end{aligned}$$

the solution of which is

$$\begin{aligned} A &= -4 \\ B &= -6 \\ C &= 4 \\ D &= 10 \end{aligned}$$

Thus

$$X(s) = \frac{-4-6s}{s^2+4} + \frac{4(s+1)+10}{(s+1)^2} = -4\frac{s}{s^2+2^2} - 3\frac{2}{s^2+s^2} + 4\frac{1}{s+1} + 10\frac{1}{(s+1)^2}$$

and

$$x(t) = -4\cos(2t) - 3\sin(2t) + 4e^{-t} + 10te^{-t}$$

15 marks for this part

(c) **Question** *Represent this solution as a sum of the free movement and the forced oscillations. Based on the form of the free movement, or otherwise, classify this system as underdamped, critically damped or overdamped.*

Answer The forced oscillations $x_p = -4\cos(2t) - 3\sin(2t)$. The free movement is $x_h = (4+10t)e^{-t}$. This is a critically damped system.

Question *Find the coordinate x of the body at the time $t = 10\pi$ s, with the precision of 5 significant figures.*

Answer $x(10\pi) \approx -4.0000$. NB: if you do it using a calculator, you should take the value of π with a rather high precision. On the other hand, this answer can be obtained without calculator, if only you care to roughly estimate the value of $e^{-10\pi}$ and see that is far too small to influence the answer in the 5 s.f. requested.

5 marks for this part

Total for this question: 25 marks
