MATH293 January 2002 exam: solutions

1. (a) **Question** *Find the general solution of the differential equation:*

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0.$$

Answer The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0,$$

and its roots are

$$\lambda_{1,2} = \{-1, -2\}$$

(positive discriminant, two real roots). Hence the basis solutions of this homogeneous equation are

$$y_1 = e^{-x}, \qquad y_2 = e^{-2x},$$

and the general solution is

$$y = \underline{C_1 e^{-x} + C_2 e^{-2x}}.$$

10 marks for this part

(b) **Question** *Using the method of undetermined coefficients, find a particular integral (particular solution) of the differential equation*

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 5\cos(x).$$

Hence write down the general solution of the equation.

Answer The free term here is a trigonometric function, the trial solution to be chosen as a combination of trigonometric functions with the same argument,

$$y_p = A\cos(x) + B\sin(x).$$

Choosing $y = y_p$, calculating y' and y'' and substituting into the nonhomogeneous equation, we obtain

$$\begin{aligned}
&y' = -A\sin(x) + B\cos(x),\\
&y'' = -A\cos(x) - B\sin(x),\\
&y'' + 3y' + 2y = -A\cos(x) - B\sin(x) - 3A\sin(x) + 3B\cos(x) + 2A\cos(x) + 2B\sin(x)\\
&= (A + 3B)\cos(x) + (B - 3A)\sin(x) = 5\cos(x)
\end{aligned}$$

Equating the coefficients at the same functions of x leads to

$$\begin{aligned}
[\cos(x)]: &\qquad A + 3B = 5,\\
[\sin(x)]: &\qquad B - 3A = 0,
\end{aligned}$$

The solution of this system is

$$A = 1/2, \quad B = 3/2,$$

and so the particular solution to the nonhomogeneous equation is

$$y_p = \frac{1}{2}\cos(x) + \frac{3}{2}\sin(x).$$

The general solution of the nonhomogeneous equation is the sum of PSNE and GSHE, that is,

$$y = \underline{\frac{1}{2}\cos(x) + \frac{3}{2}\sin(x) + C_1 e^{-x} + C_2 e^{-2x}}.$$

12 marks for this part

(c) **Question** *Suggest the form of the trial solution in the method of undetermined coefficients, for the equation*

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 5\cos(x)e^{-x}.$$

(You don't have to do the calculations here, i.e. you should leave the coefficients in the trial solution undetermined.)

Answer

$$y_p = (A\cos(x) + B\sin(x))e^{-x}.$$

3 marks for this part

Total for this question: 25 marks

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2. (a) **Question** *The function $f(x)$ is periodic, with period $p = 2L = 1$, and has the Fourier series expansion*

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(2\pi nx) + b_n\sin(2\pi nx)).$$

State the formulae for the Fourier coefficients, a_0, a_n, $n = 1, 2, \ldots$ and b_n, $n = 1, 2, \ldots$.

Answer

$$a_0 = \int_{-1/2}^{1/2} f(x)\,dx,$$

$$a_n = 2\int_{-1/2}^{1/2} f(x)\cos(2\pi nx)\,dx, \qquad n = 1, 2\ldots,$$

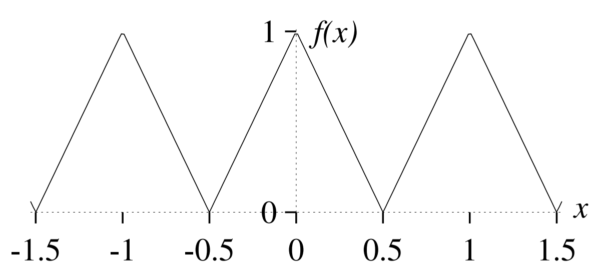
$$b_n = 2\int_{-1/2}^{1/2} f(x)\sin(2\pi nx)\,dx, \qquad n = 1, 2\ldots$$

Question *Draw the graph of $f(x)$ for $-1.5 < x < 1.5$ when*

$$f(x) = \begin{cases} 1 + 2x, & -0.5 < x < 0, \\ 1 - 2x, & 0 < x < 0.5, \end{cases}$$

and $f(x)$ is periodic with period $p = 2L = 1$.

Answer



10 marks for this part

(b) **Question** *When $f(x)$ is defined as above, briefly explain why, for all integers n,*

$$\int_{-1/2}^{1/2} f(x)\sin(2\pi nx)\,dx = 0,$$

Answer
$f(x)$ is an even function, $\sin(2\pi nx)$ is an odd function, and $\cos(2\pi nx)$ is an even function. Product of an odd function ($\sin(2\pi nx)$) by an even function ($f(x)$) is always an odd function, and an integral of an odd function over a symmetric interval $(-1,1)$ is always zero, this is why for all integer n,

$$b_n = 2\int_{-1/2}^{1/2} f(x)\sin(2\pi nx)\,\mathrm{d}x = 0.$$

Question *and why*

$$\int_{-1/2}^{1/2} f(x)\cos(2\pi nx)\,\mathrm{d}x = 2\int_{0}^{1/2} f(x)\cos(2\pi nx)\,\mathrm{d}x.$$

Answer
A product of an even function ($f(x)$) and another even function ($\cos(2\pi nx)$) is also an even function. An integral of an even function over a symmetric interval $(-1,1)$ is always double the integral over a half of that interval $(0,1)$, which is why

$$\int_{-1/2}^{1/2} f(x)\cos(2\pi nx)\,\mathrm{d}x = 2\int_{0}^{1/2} f(x)\cos(2\pi nx)\,\mathrm{d}x = 2\int_{0}^{1/2} x\cos(2\pi nx)\,\mathrm{d}x = 2I_n.$$

Question *Show that*

$$I_n = \int_0^{1/2}(1-2x)\cos(2\pi nx)\,\mathrm{d}x = \frac{1}{n\pi}\int_0^{1/2}\sin(2\pi nx)\,\mathrm{d}x, \qquad n\neq 0,$$

and hence evaluate this integral.
Answer Calculating the integral I_n by parts gives, if $n \neq 0$,

$$I_n = \int_0^{1/2}(1-2x)\cos(2\pi nx)\,\mathrm{d}x = \frac{1}{2\pi n}\int_0^{1/2}(1-2x)\,\frac{\mathrm{d}}{\mathrm{d}x}\sin(2\pi nx)\,\mathrm{d}x$$

$$= \frac{1}{2\pi n}\left((1-2x)\sin(2\pi nx)\Big|_0^{1/2} - \int_0^{1/2}\sin(2\pi nx)\frac{\mathrm{d}}{\mathrm{d}x}(1-2x)\,\mathrm{d}x\right) = \underline{\frac{1}{n\pi}\left(\int_0^{1/2}\sin(2\pi nx)\,\mathrm{d}x\right)},$$

as requested: substitution gives zero, as $\sin(0) = \sin(n\pi) = 0$.
By integrating further, we obtain

$$= \frac{1}{n\pi}\int_0^{1/2}\sin(2\pi nx)\,\mathrm{d}x = \frac{1}{2(\pi n)^2}\int_0^{1/2}\sin(2\pi nx)\,\mathrm{d}((2\pi nx)) = -\frac{1}{2(\pi n)^2}\cos(2\pi nx)\Big|_0^{1/2}$$

$$= \frac{1}{2(\pi n)^2}(1-\cos(n\pi)) = \begin{cases} 0, & n \text{ is even},\\ \dfrac{1}{n^2\pi^2}, & n \text{ is odd}, \end{cases}$$

and therefore ...
Question *Calculate a_0 and a_n and hence find the Fourier series of the function $f(x)$ defined above.*

Answer

$$a_n = 2\int\limits_{-1/2}^{1/2} f(x)\cos(2\pi n x)\,\mathrm{d}x = 4\int\limits_{0}^{1/2} f(x)\cos(2\pi n x)\,\mathrm{d}x = 4I_n = \begin{cases} 0, & n \text{ is even,} \\ \dfrac{4}{n^2\pi^2}, & n \text{ is odd.} \end{cases}$$

The constant term is:

$$a_0 = \int\limits_{-1/2}^{1/2} f(x)\,\mathrm{d}x = 2\int\limits_{0}^{1/2} (1-2x)\,\mathrm{d}x = \frac{1}{2}.$$

Thus the Fourier series is

$$f(x) = \underline{\frac{1}{2} + \frac{4}{\pi^2}\left(\cos(2\pi x) + \frac{1}{3^2}\cos(6\pi x) + \frac{1}{5^2}\cos(10\pi x) + \ldots\right)}$$

Question *By putting $x = 0$ in your Fourier series, sum the series*

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \ldots$$

Answer Substituting $x = 0$ into this Fourier seris, we obtain

$$f(0) = 1 = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos(0) + \frac{1}{3^2}\cos(0) + \frac{1}{5^2}\cos(0) + \ldots\right)$$
$$= \frac{1}{2} + \frac{4}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\right) = \frac{1}{2} + \frac{4}{\pi^2}S,$$

where

$$S = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

— is the numerical series to be summed. Since $\frac{1}{2} + \frac{4}{\pi^2}S = f(0) = 1$, we find that

$$S = \frac{1}{2}\frac{\pi^2}{4} = \underline{\frac{\pi^2}{8}}.$$

15 marks for this part

Total for this question: 25 marks

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3. (a) **Question** *The Fourier transform $\tilde{f}(w)$ of a function $f(t)$ is defined by*

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t)e^{-iwt}\,\mathrm{d}t.$$

Give the formula, by which $f(t)$ can be found if its Fourier transform $\tilde{f}(w)$ is known.

Answer

$$f(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \tilde{f}(w)e^{iwt}\,\mathrm{d}w.$$

General structure , correct coefficient , correct limits , correct sign in the exponent .

6 marks for this part

(b) **Question** *Show that if $f(t)$ is given by*

$$f(t) = \begin{cases} 0, & t \le -1/2, \\ 1+2t, & -1/2 < t < 0, \\ 1-2t, & 0 < t < 1/2, \\ 0, & t > 1/2, \end{cases}$$

its Fourier transform is

$$\tilde{f}(w) = \frac{4}{\sqrt{2\pi}} \frac{1-\cos(w/2)}{w^2}$$

Answer For the given $f(t)$, the Fourier transform is

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt}\, \mathrm{d}t$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-1/2}^{0} (1+2t)e^{-iwt}\, \mathrm{d}t + \int_{0}^{1/2} (1-2t)e^{-iwt}\, \mathrm{d}t \right)$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1/2} (1-2t)\cos(wt)\, \mathrm{d}t$$

(this step is optional, straightforward calculation of the two integrals fully acceptable)

$$= \frac{2}{\sqrt{2\pi}} \left(\frac{1}{w}\sin(wt)\,(1-2t) + \frac{2}{w}\int \sin(wt)\, \mathrm{d}t \right) \Bigg|_{0}^{1/2}$$

$$= \frac{2}{\sqrt{2\pi}} \left(\frac{1}{w}\sin(wt)\,(1-2t) - \frac{2}{w^2}\cos(wt) \right) \Bigg|_{0}^{1/2}$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left(\frac{1}{w}\sin(w/2)\cdot 0 - \frac{2}{w^2}\cos(w/2) \right) - \left(\frac{1}{w}\sin(0)\cdot 1 - \frac{2}{w^2}\cos(0) \right) \right]$$

$$= \frac{4}{\sqrt{2\pi}} \frac{1-\cos(w/2)}{w^2}$$

as requested.

15 marks for this part

(c) **Question** *Using the result of parts (a) and (b), write down the integral which represents $f(t)$ defined as above.*

Answer Inverse F.t.:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(w) e^{iwt}\, \mathrm{d}w = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1-\cos(w/2)}{w^2} e^{iwt}\, \mathrm{d}w$$

Question

By putting $t = 0$ in this result and substituting $x = w/2$, evaluate

$$\int_{-\infty}^{\infty} \frac{1-\cos(x)}{x^2}\, \mathrm{d}x.$$

Answer For $t = 0$,

$$f(0) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos(w/2)}{w^2} e^{iw\cdot 0}\,\mathrm{d}w = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos(w/2)}{w^2}\,\mathrm{d}w$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \frac{1-\cos(x)}{(2x)^2}\,2\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\infty} \frac{1-\cos(x)}{x^2}\,\mathrm{d}x = f(0) = 1$$

hence

$$\int_{-\infty}^{\infty} \frac{1-\cos(x)}{x^2}\,\mathrm{d}x = \pi$$

4 marks for this part

Total for this question: 25 marks

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4. (a) **Question** *Find the function of t whose Laplace transform is:*

$$\frac{1}{(s+1)(s+4)}$$

Answer By cover-up rule,

$$F(s) = \frac{1}{(s+1)(s+4)} = \frac{1}{(-1)+4}\cdot\frac{1}{s+1} + \frac{1}{(-4)+1}\cdot\frac{1}{s+4}$$

$$= \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}$$

(any other valid method equally acceptable)

$$f(t) = \mathcal{L}^{-1}(F(s)) = \boxed{\frac{1}{3}\left(e^{-t} - e^{-4t}\right)}$$

4 marks for this part

(b) **Question** *Find the function of t whose Laplace transform is:*

$$\frac{s+7}{s^2+6s+13}$$

Answer

$$F(s) = \frac{s+7}{s^2+6s+13} = \frac{s+7}{s^2+6s+9+4} = \frac{(s+3)+4}{(s+3)^2+2^2}$$

$$= \frac{(s+3)}{(s+3)^2+2^2} + 2\frac{2}{(s+3)^2+2^2}$$

$$f(t) = \mathcal{L}^{-1}\frac{(s+3)}{(s+3)^2+2^2} + 2\mathcal{L}^{-1}\frac{2}{(s+3)^2+2^2} = e^{-3t}\cos(2t) + 2e^{-3t}\sin(2t)$$

8 marks for this part

(c) **Question** *Find, **using the Laplace Transform**, the solution of the differential equation*

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 9y = 16\cos(5t),$$

which satisfies initial conditions

$$y(0) = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t}(0) = 3.$$

Answer Let $\mathcal{L}y = Y$, then with account of the initial conditions,

$$\mathcal{L}y'' = s^2Y - sy(0) - y'(0) =$$
$$s^2Y - 3,$$

and the subsidiary equation is

$$s^2Y - 3 + 9Y = \frac{16s}{s^2+25}$$

Its solution is:

$$Y = \frac{16s}{(s^2+9)(s^2+25)} + \frac{3}{s^2+9}$$
$$= \frac{s}{s^2+9} - \frac{s}{s^2+25} + \frac{3}{s^2+9}.$$

Thus

$$y = \mathcal{L}^{-1}[Y] = \underline{-\cos(5t) + \cos(3t) + \sin(3t).}$$

Question *Check your solution by substituting $y(t)$ into the differential equation and initial conditions.*

Answer

$$y(0) = -1 + 1 + 0 = 0 \quad \checkmark$$
$$y'(t) = 5\sin(5t) - 3\sin(3t) + 3\cos(3t),$$
$$y'(0) = 5\cdot 0 - 3\cdot 0 + 3\cdot 1 = 3 \quad \checkmark$$
$$y''(t) = 25\cos(5t) - 9\cos(3t) - 9\sin(3t),$$
$$y'' + 9y = 25\cos(5t) - 9\cos(3t) - 9\sin(3t) + 9(-\cos(5t) + \cos(3t) + \sin(3t))$$
$$= (25-9)\cos(5t) + (-9+9)\cos(3t) + (-9+9)\sin(3t) = 16\cos(5t) \quad \checkmark$$

13 marks for this part

Total for this question: 25 marks

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5. **Question** *The functions $x(t)$ and $y(t)$ satisfy the differential equations*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 16x - 10y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 26x - 16y$$

and the initial conditions

$$x(0) = 2 \quad \text{and} \quad y(0) = 3.$$

(a) **Question** *Show that X and Y, the Laplace transforms of $x(t)$ and $y(t)$, are given by*

$$X(s) = \frac{2s+2}{s^2+4}, \qquad Y(s) = \frac{3s+4}{s^2+4}.$$

Answer Let $X = \mathcal{L}x$, $Y = \mathcal{L}y$. The subsidiary system is

$$\begin{aligned} sX - 2 &= 16X - 10Y, \\ sY - 3 &= 26X - 16Y. \end{aligned}$$

which has the solutions as requested

$$\begin{aligned} X(s) &= \frac{2s+2}{s^2+4}, \\ Y(s) &= \frac{3s+4}{s^2+4} \end{aligned}$$

(any valid method acceptable)

Question *Hence find $x(t)$ and $y(t)$.*

Answer

$$\begin{aligned} x(t) &= 2\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] + \mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] \\ &= \underline{2\cos(2t) + \sin(2t)} \\ y(t) &= 3\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] + 2\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] \\ &= \underline{3\cos(2t) + 2\sin(2t)} \end{aligned}$$

16 marks for this part

(b) **Question** *Verify your solution by substituting $x(t)$, $y(t)$ into the differential equations and initial conditions.*

Answer *attempted =*

$$x(0) = 2\cos(0) + \sin(0) = 2, \quad \checkmark$$

$$y(0) = 3\cos(0) + 4\sin(0) = 3, \quad \checkmark$$

$$x' = -4\sin(2t) + 2\cos(2t);$$

$$16x - 10y = 32\cos(2t) + 16\sin(2t) - 30\cos(2t) - 20\sin(2t)) = 2\cos(2t) - 4\sin(2t) = x' \quad \checkmark$$

$$y' = -6\sin(2t) + 4\cos(2t);$$

$$26x - 16y = 52\cos(2t) + 26\sin(2t) - 48\cos(2t) - 32\sin(2t) == 4\cos(2t) - 6\sin(2t) = y' \quad \checkmark$$

9 marks for this part

Total for this question: 25 marks
