JANUARY 2002 EXAMINATIONS

Degree of Bachelor of Engineering : Year 2
Degree of Bachelor of Engineering : Year 3
Degree of Master of Engineering : Year 2

ENGINEERING MATHEMATICS I

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FOUR questions.
Only the best FOUR answers will be counted.

1. (a) Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0.$$

(10 marks)

(b) Using the method of undetermined coefficients, find a particular integral (particular solution) of the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 5\cos(x).$$

Hence write down the general solution of the equation.

(12 marks)

(c) Suggest the form of the trial solution in the method of undetermined coefficients, for the equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 5\cos(x)e^{-x}.$$

(You don't have to do the calculations here, i.e. you should leave the coefficients in the trial solution undetermined.)

(3 marks)

2. (a) The function $f(x)$ is periodic, with period $p = 2L = 1$, and has the Fourier series expansion

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(2\pi n x) + b_n \sin(2\pi n x)).$$

State the formulae for the Fourier coefficients, a_0, a_n, $n = 1, 2, \dots$ and b_n, $n = 1, 2, \dots$.

Draw the graph of $f(x)$ for $-1.5 < x < 1.5$ when

$$f(x) = \begin{cases} 1 + 2x, & -0.5 < x < 0, \\ 1 - 2x, & 0 < x < 0.5, \end{cases}$$

and $f(x)$ is periodic with period $p = 2L = 1$.

(10 marks)

(b) When $f(x)$ is defined as above, briefly explain why, for all integers n, $\int_{-1/2}^{1/2} f(x) \sin(2\pi n x)\, \mathrm{d}x = 0$, and why

$$\int_{-1/2}^{1/2} f(x) \cos(2\pi n x)\, \mathrm{d}x = 2 \int_{0}^{1/2} f(x) \cos(2\pi n x)\, \mathrm{d}x.$$

Show that

$$I_n = \int_0^{1/2} (1 - 2x) \cos(2\pi n x)\, \mathrm{d}x = \frac{1}{n\pi} \int_0^{1/2} \sin(2\pi n x)\, \mathrm{d}x, \qquad n \neq 0,$$

and hence evaluate this integral. Calculate a_0 and a_n and hence find the Fourier series of the function $f(x)$ defined above.

By putting $x = 0$ in your Fourier series, sum the series

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots$$

(15 marks)

3. (a) The Fourier transform $\tilde{f}(w)$ of a function $f(t)$ is defined by

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt} \, \mathrm{d}t.$$

Give the formula, by which $f(t)$ can be found if its Fourier transform $\tilde{f}(w)$ is known.

(6 marks)

(b) Show that if $f(t)$ is given by

$$f(t) = \begin{cases} 0, & t \leq -1/2, \\ 1 + 2t, & -1/2 < t < 0, \\ 1 - 2t, & 0 < t < 1/2, \\ 0, & t > 1/2, \end{cases}$$

its Fourier transform is

$$\tilde{f}(w) = \frac{4}{\sqrt{2\pi}} \frac{1 - \cos(w/2)}{w^2}$$

(15 marks)

(c) Using the result of parts (a) and (b), write down the integral which represents $f(t)$ defined as above.

By putting $t = 0$ in this result and substituting $x = w/2$, evaluate

$$\int_{-\infty}^{\infty} \frac{1 - \cos(x)}{x^2} \, \mathrm{d}x.$$

(4 marks)

4. (a) Find the function of t whose Laplace transform is:

$$\frac{1}{(s+1)(s+4)}$$

(4 marks)

(b) Find the function of t whose Laplace transform is:

$$\frac{s+7}{s^2+6s+13}$$

(8 marks)

(c) Find, **using the Laplace Transform**, the solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 9y = 16\cos(5t),$$

which satisfies initial conditions

$$y(0) = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t}(0) = 3.$$

Check your solution by substituting $y(t)$ into the differential equation and initial conditions.

(13 marks)

Note: a table of standard Laplace transforms is available on page 7.

5. The functions $x(t)$ and $y(t)$ satisfy the differential equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 16x - 10y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 26x - 16y \end{aligned}$$

and the initial conditions

$$x(0) = 2 \quad \text{and} \quad y(0) = 3.$$

(a) Show that X and Y, the Laplace transforms of $x(t)$ and $y(t)$, are given by

$$X(s) = \frac{2s+2}{s^2+4}, \qquad Y(s) = \frac{3s+4}{s^2+4}.$$

Hence find $x(t)$ and $y(t)$.

(16 marks)

(b) Verify your solution by substituting $x(t)$, $y(t)$ into the differential equations and initial conditions.

(9 marks)

Note: a table of standard Laplace transforms is available on page 7.

Table of Laplace transforms

$f(t)$ (original)	$F(s)$ (image)	$f(t)$ (original)	$F(s)$ (image)
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
t	$\frac{1}{s^2}$	te^{at}	$\frac{1}{(s-a)^2}$
t^2	$\frac{2}{s^3}$	t^2e^{at}	$\frac{2}{(s-a)^3}$
t^n	$\frac{n!}{s^{n+1}}$	t^ne^{at}	$\frac{n!}{(s-a)^{n+1}}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$\frac{t\sin(\omega t)}{2\omega}$	$\frac{s}{(s^2+\omega^2)^2}$	$\frac{t\sin(\omega t)}{2\omega}e^{at}$	$\frac{s-a}{((s-a)^2+\omega^2)^2}$
$\frac{\sin(\omega t)-\omega t\cos(\omega t)}{2\omega^2}$	$\frac{\omega}{(s^2+\omega^2)^2}$	$\frac{\sin(\omega t)-\omega t\cos(\omega t)}{2\omega^2}e^{at}$	$\frac{\omega}{((s-a)^2+\omega^2)^2}$
$y(t)$	$Y(s)$	$e^{at}y(t)$	$Y(s-a)$
$\frac{\mathrm{d}y(t)}{\mathrm{d}t}$	$sY(s)-y(0)$	$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}$	$s^2Y(s)-sy(0)-y'(0)$