PAPER CODE NO.
MATH285

THE UNIVERSITY
of LIVERPOOL

SEPTEMBER 2004 EXAMINATIONS

Bachelor of Science : Year 1
Bachelor of Science : Year 2
Master of Physics : Year 2

MECHANICS AND RELATIVITY

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES: Attempt FOUR questions only.

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1. (a) A particle of mass m is released from rest at time $t = 0$ and allowed to fall under the influence of gravity. The particle is also subject to a resistive force equal to k times the speed of the particle.

Show that if the position x of the particle is measured downwards from the point at which it is released, then the equation of motion of the particle is

$$\frac{dv}{dt} = g - \frac{k}{m}v \ ,$$

where g is the gravitational acceleration.

[3 marks]

Verify by differentiation that

$$v(t) = \frac{mg}{k} + Ae^{-kt/m}$$

is a solution of the differential equation for v for any value of the constant A. Given that $v = 0$ at $t = 0$, show that $A = -mg/k$. Find an expression for the terminal velocity of the particle.

[6 marks]

Calculate the displacement of the particle $x(t)$ for an arbitrary time t.

[6 marks]

(b) A particle with mass m is launched from the origin along the positive x-axis with initial velocity $\dot{x} = 2$. The particle is subject to the force

$$F(x) = k(3 - x) \ ,$$

where $k > 0$ is a constant.

Show that the potential energy function $V(x)$ for the force, assuming that $V(x) = 0$ at $x = 0$, is given by

$$V(x) = k\left(\frac{x^2}{2} - 3x\right) \ .$$

Hence write down the particle's total energy as a function of $\dot{x}$ and x.

[4 marks]

If $m = 1$ and $k = 4$, use the initial conditions for x and $\dot{x}$ to show that $E = 2$. Then use the fact that the energy is a constant to find $\dot{x}^2$ as a function of x. Hence calculate the position of both of the turning points where the particle changes direction.

[6 marks]

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2. A particle of mass m and charge q starts at the origin and is launched at $t = 0$ with an initial speed v_o in the positive y direction. The particle then moves under the influence of a magnetic field given by

$$\mathbf{B} = (0, 0, B) \ .$$

The equation of motion for the particle is

$$m\frac{d^2\mathbf{r}}{dt^2} = q\mathbf{v} \times \mathbf{B} \ ,$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the velocity of the particle.

Show that the equations of motion for the particle can be written as

$$m\ddot{x} = qB\dot{y} \ ,$$

$$m\ddot{y} = -qB\dot{x} \ ,$$

$$m\ddot{z} = 0 \ .$$

[4 marks]

By integrating with respect to time and using the initial conditions, show that the velocities can be written as

$$\dot{x} = \omega y \ ,$$

$$\dot{y} = -\omega x + v_o \ ,$$

$$\dot{z} = 0 \ ,$$

where $\omega = qB/m$.

[6 marks]

Hence show that the equation for $\ddot{z}$ can be written as

$$\ddot{y} = -\omega^2 y \ .$$

[1 mark]

Write down the general solution for this equation, and hence show that $y(t)$ is given by

$$y(t) = \frac{v_o}{\omega} \sin(\omega t) \ .$$

Using this result find $\dot{x}(t)$ and so $x(t)$.

[10 marks]

Deduce that the trajectory is a circle and find the radius and the position of the centre. Sketch the trajectory.

[4 marks]

3. A projectile is launched at $t = 0$ with initial velocity $\mathbf{v}_o$ and initial displacement $\mathbf{r}_o$ relative to an origin fixed on the surface of the Earth. The equation of motion in the local coordinates attached to the Earth's surface is

$$\frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{k} - 2\boldsymbol{\omega} \times \mathbf{v} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \ ,$$

where g is the gravitational acceleration, $\boldsymbol{\omega}$ is the constant angular velocity of the Earth, and $\mathbf{r}$ and $\mathbf{v}$ represent the projectile's displacement and velocity respectively. $\mathbf{k}$ is the unit coordinate vector pointing straight up from the Earth's surface, $\mathbf{i}$ is 'East' and $\mathbf{j}$ is 'North'.

(a) Write down the condition between $|\boldsymbol{\omega} \times \mathbf{r}|$ and $|\mathbf{v}|$ that must be satisfied in order that the equation of motion can be generally approximated by

$$\frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{k} - 2\boldsymbol{\omega} \times \mathbf{v} \qquad (*) \ ,$$

where $-2\boldsymbol{\omega}\times\mathbf{v}$ is due to the Coriolis force.

By integrating this equation, find an expression for $\mathbf{v}(t)$ in terms of $\boldsymbol{\omega}$, $\mathbf{r}$ and t.

[6 marks]

Substitute your expression for the velocity back into equation $(*)$ and hence show that the acceleration is, to leading order in $\boldsymbol{\omega}$ (i.e. ignoring terms with two or more factors of $\boldsymbol{\omega}$),

$$\frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{k} + 2gt\boldsymbol{\omega} \times \mathbf{k} - 2\boldsymbol{\omega} \times \mathbf{v}_o \ .$$

Hence show that the displacement at time t is

$$\mathbf{r}(t) = \mathbf{r}_o + \mathbf{v}_o t - \frac{1}{2}gt^2\mathbf{k} - t^2\boldsymbol{\omega} \times \mathbf{v}_o + \frac{1}{3}gt^3\boldsymbol{\omega} \times \mathbf{k} \ .$$

[10 marks]

(b) A projectile is fired horizontally at $t = 0$ with a velocity of 50 ms^{-1} due East, from a point on the Earth's surface with latitude 45^o and 100m above the ground.

Show that

$$\boldsymbol{\omega} = \frac{1}{\sqrt{2}}(\mathbf{j} + \mathbf{k})\,\omega \ ,$$

where $\omega = |\boldsymbol{\omega}|$. Hence show that the deviation of the particle due to the Coriolis force terms in the solution for $\mathbf{r}(t)$ (i.e. the terms involving $\boldsymbol{\omega}$) is given by

$$\Delta\mathbf{r} = -50t^2\omega\frac{1}{\sqrt{2}}(\mathbf{j}-\mathbf{k}) + \frac{gt^3}{3}\frac{\omega}{\sqrt{2}}\mathbf{i} \, .$$

Calculate the deviation of the projectile to the North after 3 seconds of flight. [You may use $g = 9.81\text{ms}^{-2}$ and $\omega = |\boldsymbol{\omega}| = 7.27 \times 10^{-5}\text{rad s}^{-1}$.]

[9 marks]

4. (a) Show that the moment of inertia of a uniform circular lamina of mass M and radius a about an axis through its centre of mass and perpendicular to the plane of the lamina is given by

$$I_{zz} = \frac{Ma^2}{2} \ .$$

State the Perpendicular Axes theorem for a lamina and hence or otherwise show that the moment of inertia of the circular lamina about an axis through the centre of mass and parallel to the plane of the lamina is given by

$$I_{xx} = \frac{Ma^2}{4} \ .$$

[9 marks]

(b) Explain briefly the meaning of the terms *principal axes* and *principal moments of inertia*. Describe a set of principal axes for the circular lamina above. What are the principal moments of inertia about these axes?

[5 marks]

(c) The components of the angular velocity of a rigid body rotating under the action of a torque $\mathbf{N}$ satisfy the equations:

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = N_1 \ ,$$

$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = N_2 \ ,$$

$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = N_3 \ ,$$

where the components of $\boldsymbol{\omega}$ are taken along the principal axes of the rigid body. I_1, I_2 and I_3 are the principal moments of inertia and N_1, N_2 and N_3 are the components of $\mathbf{N}$ about these principal axes.

The circular lamina described in (a) is rotating with constant angular velocity with respect to the principal axes about an axis through its centre of mass inclined at an angle of 45^o to the normal to the plane of the lamina. [You may assume that the angular velocity has non-zero components along the 1 and 3 principal axes, and that the component along the 2 principal axis is zero, where the 1 and 2 axes are in the plane of the lamina.]

Show that the the components of the angular velocity along the principal axes are given by

$$\boldsymbol{\omega} = \left(\frac{\omega}{\sqrt{2}}, 0, \frac{\omega}{\sqrt{2}}\right) ,$$

where $\omega = |\boldsymbol{\omega}|$. State the components of the angular momentum. Find the moment $\mathbf{N}$ which is needed to keep the angular velocity constant (i.e. $\dot{\boldsymbol{\omega}} = 0$).

[11 marks]

5. (a) An object is observed at a point x^a. The same object is observed to be at a point x'^a as seen from a second reference frame, which is moving with velocity v along the x-axis with respect to the first frame.

The relation between these is given by the Lorentz transformation

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

If the squared norm of $R = (x^0, x^1, x^2, x^3)$ is defined to be

$$R^2 = g_{ab}x^a x^b \ ,$$

where $g_{ab} = diag(1, -1, -1, -1)$, show that $R^2 = R'^2$, where $R' = (x'^0, x'^1, x'^2, x'^3)$. What property of R^2 is demonstrated by this?

[10 marks]

(b) A particle of rest mass m_o moves along the x-axis of a coordinate system with speed u. Write down expressions for its relativistic momentum $\mathbf{p}$ and its total energy E in terms of m_o, u and $\gamma(u)$. [You may use the 4-momentum relation $\mathbf{P} \equiv (E/c, \mathbf{p}) = m_o\mathbf{U}$ and $\mathbf{U} = \gamma(u)\,(c, \mathbf{u})$.] By substituting these expressions prove that

$$E^2 - \mathbf{p}^2c^2 = m_o^2c^4 \ .$$

[5 marks]

(c) A particle of rest mass M at rest in an inertial frame decays into two particles with rest masses m_1 and m_2 respectively. Show that the energy of the particle of rest mass m_1 is

$$\frac{(M^2 + m_1^2 - m_2^2)\,c^2}{2M} \ .$$

[10 marks]