PAPER CODE NO.
**MATH285**

THE UNIVERSITY
*of* LIVERPOOL

# JANUARY 2006 EXAMINATIONS

Bachelor of Science : Year 2
Master of Physics : Year 2

## MECHANICS AND RELATIVITY

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES:
Full marks can be obtained for complete answers to FOUR questions.
Only the best FOUR answers will be counted.

THE UNIVERSITY
*of* LIVERPOOL

1. (a) A particle of mass $m$ moving along the $x$-axis is acted on by a force

$$F(t) = Ae^{-\alpha t} + Bt$$

at time $t$, where $A$, $\alpha$ and $B$ are constants.

Using Newton's laws, calculate the velocity $\dot{x}(t)$ and the position $x(t)$ given that $\dot{x} = 0$ and $x = 0$ when $t = 0$.

[**10 marks**]

(b) A particle with mass $m$ is moving on the $x$-axis under the influence of a force given by

$$F(x) = -A \sin kx$$

where $A$ and $k$ are positive constants. Calculate the potential energy function $V(x)$ for this force, assuming that $V(x) = 0$ at $x = 0$.

[**6 marks**]

Draw a sketch of this potential energy $V(x)$.

[**3 marks**]

If the particle is initially at $x = 0$ with a velocity $v$ describe in words how you would expect it to move for small initial velocity and for large velocity. Which velocity marks the change-over between the two behaviours?

[**6 marks**]

2. A particle of mass $m$ moves in three dimensions in the potential

$$V(x, y, z) = \tfrac{1}{2}Kx^2 + mgz$$

where $K$ and $g$ are constants.

Calculate the force acting on the particle. Use this to show that the equations of motion for the particle are

$$\begin{aligned} \ddot{x} &= -\frac{K}{m}x \\ \ddot{y} &= 0 \\ \ddot{z} &= -g \; . \end{aligned}$$

Find the general solutions of the equations for $x, y$ and $z$.

**[12 marks]**

At $t = 0$ the particle's position and velocity are

$$\begin{aligned} \mathbf{r} &= (a, 0, 0) \\ \dot{\mathbf{r}} &= (0, u, 0) \; . \end{aligned}$$

Find $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$.

Use this solution to calculate the potential energy, the kinetic energy and the total energy at time $t$. Comment on the physical reason for your answers.

**[13 marks]**

3. A particle of mass $m$ moving in a central potential $V(r)$ has two conserved quantities

$$\begin{aligned} J &= mr^2 \frac{d\theta}{dt} \,, \\ \text{and} \quad E &= \tfrac{1}{2} m \left( \frac{dr}{dt} \right)^2 + \frac{J^2}{2mr^2} + V(r) \,. \end{aligned}$$

Here $r$ and $\theta$ are the position of the particle in polar coordinates.

Explain briefly the physical meaning of $J$ and $E$.

[**4 marks**]

Use the chain rule to prove

$$\frac{dr}{dt} = \frac{J}{mr^2} \left( \frac{dr}{d\theta} \right)$$

and hence show that

$$E = \frac{J^2}{2mr^4} \left\{ \left( \frac{dr}{d\theta} \right)^2 + r^2 \right\} + V(r) \,. \tag{1}$$

[**10 marks**]

A particle moving in a central potential follows the path

$$r = C \cosh(b\theta) \,.$$

Use the result (1) to find the potential $V(r)$ in terms of $r$ and the constants in the problem, $E, J, b$ and $C$.

[**11 marks**]

[**Hint:** The results

$$\frac{d}{dx} \cosh x = \sinh x$$

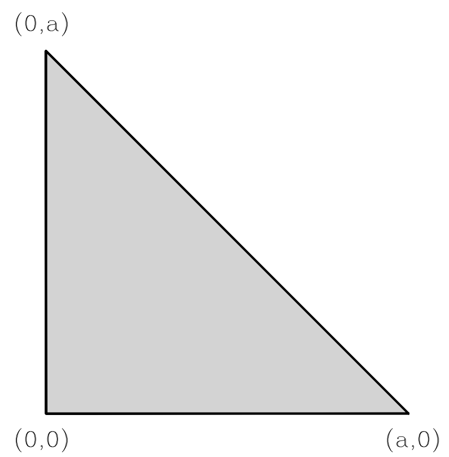
and

$$\sinh^2 x = \cosh^2 x - 1$$

may prove useful.]

4. (a) Find the centre of mass of the thin uniform right-angled triangular slab below. The mass of the lamina is $M$ and the short sides of the triangle have length $a$.

**[4 marks]**




Find the moment of inertia of the triangle if it is rotating about an axis perpendicular to the plane of the triangle, passing through the vertex $(0,0)$.

**[5 marks]**

State the Parallel Axes theorem for moments of inertia, and hence or otherwise find the moment of inertia of the slab if it is rotating about an axis passing through the centre of mass, perpendicular to the plane of the lamina.

**[6 marks]**

(b) A ring and a solid sphere, both of mass $M$ and radius $R$, are released from rest at the top of a ramp of height $h$ and roll down to the base. Use the conservation of energy to calculate their final speeds.

Calculate the angular momentum of each object when it reaches the bottom of the ramp.

**[10 marks]**

[You may use the results

$$I = MR^2 \quad \text{for a ring,}$$
$$I = \tfrac{2}{5}MR^2 \quad \text{for a solid sphere. ]}$$

THE UNIVERSITY *of* LIVERPOOL

5. An inertial frame $S'$ is moving with velocity $v$ along the $x$-axis relative to another frame $S$. The Lorentz transformation converting the coordinates in $S$ to coordinates in $S'$ is

$$t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z$$

where $$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and $c$ is the speed of light.

(a) In frame $S$ a particle moves with the constant three-velocity $\mathbf{u} = (u_x, 0, 0)$. If it is at the origin at $t = 0$, calculate its coordinates in the frame $S$ at time $t$.

Transform these coordinates into the frame $S'$, moving with relative velocity $v$ along the $x$-axis. Use your result to calculate the particle's three-velocity $\mathbf{u}'$ as observed in the frame $S'$.

[**8 marks**]

If $u_x = c$ find $\mathbf{u}'$ and comment on the physical significance of your answer.

[**5 marks**]

(b) State the relativistic expression for the energy of a particle with momentum $p$ and rest mass $m$.

What is the rest mass of a photon?

A photon of energy $E_\gamma$ is absorbed by a stationary nucleus of rest mass $M_1$, creating an excited nucleus with rest mass $M_2$ moving at velocity $v$.

Use the conservation of energy and momentum to show that

$$v = \frac{E_\gamma c}{M_1 c^2 + E_\gamma}\ .$$

Find the rest mass of the excited nucleus.

[**12 marks**]