PAPER CODE NO.
MATH285

THE UNIVERSITY
of LIVERPOOL

JANUARY 2005 SOLUTIONS

Bachelor of Science : Year 1
Bachelor of Science : Year 2
Master of Physics : Year 2

MECHANICS AND RELATIVITY

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES: Attempt FOUR questions only.

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1. (a) *A particle of mass m moving along the x-axis is acted on by a force*

$$F(t) = A\cos\alpha t - Bt^2$$

at time t, where A, α and B are constants.

Using Newton's law, calculate the velocity $\dot{x}(t)$ and the position $x(t)$ given that $\dot{x} = 0$ and $x = 0$ when $t = 0$.

Because we know the force as a function of time we can solve this equation by integrating with respect to t. (This would not work if the force depends on position instead of time.)

Starting from Newton's second law

$$F = ma = m\ddot{x}$$

$$\Rightarrow \qquad \ddot{x} = \frac{F}{m} = \frac{A}{m}\cos\alpha t - \frac{B}{m}t^2$$

Integrate once with respect to time to find the velocity $\dot{x}$.

$$\dot{x} = \int\left(\frac{A}{m}\cos\alpha t - \frac{B}{m}t^2\right)\,dt = \frac{A}{m\alpha}\sin\alpha t - \frac{B}{3m}t^3 + C$$

(don't forget C, the constant of integration!).

To find C use the initial condition $\dot{x}(0) = 0, \quad \Rightarrow \quad C = 0$.

$$\dot{x} = \frac{A}{m\alpha}\sin\alpha t - \frac{B}{3m}t^3$$

Integrate again to find $x(t)$.

$$x = -\frac{A}{m\alpha^2}\cos\alpha t - \frac{B}{12m}t^4 + D$$

Initial condition $x(0) = 0$

$$\Rightarrow \quad D = \frac{A}{m\alpha^2}$$

so

$$x = \frac{A}{m\alpha^2}(1 - \cos\alpha t) - \frac{B}{12m}t^4\ .$$

[10 marks]

(b) *A particle with mass m is moving on the x-axis under the influence of a force given by*

$$F(x) = A - Kx$$

where A and K are constants. Calculate the potential energy function $V(x)$ for this force, assuming that $V(x) = 0$ at $x = 0$.
Hence write down the particle's total energy if it is at position x and has velocity v.

Starting point is the equation

$$F(x) = -\frac{d}{dx}V(x) \qquad \Leftrightarrow \qquad V(x) = -\int F(x)\,dx$$

$$V(x) = -\int (A - Kx)dx = -Ax + \tfrac{1}{2}Kx^2 + C$$

From $V(0) = 0$ we get $C = 0$ so

$$\underline{\underline{V(x) = -Ax + \tfrac{1}{2}Kx^2}}$$

Total energy:

$$E = K.E. + P.E. = \tfrac{1}{2}mv^2 + V(x) = \underline{\underline{\tfrac{1}{2}mv^2 - Ax + \tfrac{1}{2}Kx^2}}$$

[**7 marks**]

Take the mass of the particle to be 1 kg, and the values of the constants to be $A = 2$ N and $K = 2$ N m^{-1}. The particle is launched from $x = 0$ with an initial velocity of 2 m s^{-1}.
What is its total energy?

When $x = 0$ the potential energy V is 0, so the total energy is the same as the kinetic energy,

$$E = \tfrac{1}{2}mv_0^2 + V(0) = \tfrac{1}{2} \times 1 \times 2^2 + 0 = 2 \text{ J}.$$

Express the velocity v as a function of x. Use this to calculate the positions of both turning points of the particle.

The total energy, K.E. + P.E., doesn't change, it is always 2 J. So we have

$$\tfrac{1}{2}mv^2 + V(x) = 2$$

$$\Rightarrow \qquad v^2 = \frac{2}{m}(2 - V(x)) = \frac{2}{m}\left(2 + Ax - \tfrac{1}{2}Kx^2\right)$$

$$\Rightarrow \qquad v^2 = 2\left(2 + 2x - x^2\right)$$

putting in the values for all the constants.

$$\underline{\underline{v = \pm\sqrt{4 + 4x - 2x^2}}}$$

The turning points are the places where $v = 0$, i.e. where the kinetic energy is 0 and all the energy is potential energy. They are at

$$4 + 4x - 2x^2 = 0 \quad \Rightarrow \quad \underline{\underline{x = \left(1 \pm \sqrt{3}\right) \text{ m}}}$$

[**8 marks**]

2. *A particle of mass m moves in three dimensions in the potential*

$$V(x,y,z) = \tfrac{1}{2}m\omega^2(x^2+y^2+z^2),$$

where ω is a constant.

Calculate the force acting on the particle.

To get a force from a potential we need to take the gradient

$$\mathbf{F} = -\nabla\, V\ .$$

Writing out the three components separately gives

$$\begin{aligned} F_x &= -\frac{\partial}{\partial x}V = -m\omega^2 x \\ F_y &= -\frac{\partial}{\partial y}V = -m\omega^2 y \\ F_z &= -\frac{\partial}{\partial z}V = -m\omega^2 z \end{aligned}$$

Use this to show that the equations of motion for the particle are

$$\begin{aligned} \ddot{x} &= -\omega^2 x \\ \ddot{y} &= -\omega^2 y \\ \ddot{z} &= -\omega^2 z\ . \end{aligned}$$

From Newton's Law

$$\begin{aligned} & \mathbf{F} = m\mathbf{a} \\ \Rightarrow\quad & (F_x, F_y, F_z) = m(\ddot{x}, \ddot{y}, \ddot{z}) \\ \Rightarrow\quad & (-m\omega^2 x, -m\omega^2 y, -m\omega^2 z) = m(\ddot{x}, \ddot{y}, \ddot{z}) \end{aligned}$$

Writing the three components separately gives the three equations of motion

$$\begin{aligned} \ddot{x} &= -\omega^2 x \\ \ddot{y} &= -\omega^2 y \\ \ddot{z} &= -\omega^2 z \end{aligned}$$

just as the question told us .

Find the general solutions of the equations for x, y and z.

Luckily each of the equations only involves one coordinate, so they can be solved separately. Do the x equation first. This is

$$\ddot{x} + \omega^2 x = 0$$

which is an equation you should recognise. You can solve it by making the guess $x = \exp(rt)$ and solving for r. The value of r is $i\omega$ so the general solution for x is

$$\underline{\underline{x = A\cos\omega t + B\sin\omega t\ .}}$$

The other equations work the same way, the complete solution is

$$\begin{aligned} x &= A\cos\omega t + B\sin\omega t \\ y &= C\cos\omega t + D\sin\omega t \\ z &= E\cos\omega t + F\sin\omega t \end{aligned}$$

[12 marks]

At $t = 0$ the particle's position and velocity are

$$\begin{aligned} \mathbf{r} &= (a, 0, 0) \\ \dot{\mathbf{r}} &= (0, u, 0)\ . \end{aligned}$$

Find $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$.

From the general solution the position at $t = 0$ is

$$(x, y, z) = (A, C, E)$$

For this to match the initial condition $\mathbf{r} = (a, 0, 0)$ we must have

$$\underline{\underline{A = a, \qquad C = 0, \qquad E = 0\ .}}$$

Similarly, the general solution says that the velocity at $t = 0$ is

$$(\dot{x}, \dot{y}, \dot{z}) = (B\omega, D\omega, F\omega)$$

and to match the initial condition $\dot{\mathbf{r}} = (0, u, 0)$ we must have

$$\underline{\underline{B = 0, \qquad D = \frac{u}{\omega}, \qquad E = 0\ .}}$$

Putting these values for the constants into the solution gives

$$\begin{aligned} \mathbf{r}(t) &= (a\cos\omega t,\ \frac{u}{\omega}\sin\omega t, 0) \\ \dot{\mathbf{r}}(t) &= (-a\omega\sin\omega t,\ u\cos\omega t, 0) \end{aligned}$$

Use this solution to calculate the potential energy, the kinetic energy and the angular momentum at time t. Comment on the physical reason for your answers.

The potential energy is

$$\text{P.E.} = V = \tfrac{1}{2}m\omega^2(x^2+y^2+z^2) = \underline{\underline{\tfrac{1}{2}m\omega^2a^2\cos^2\omega t + \tfrac{1}{2}mu^2\sin^2\omega t}}$$

The kinetic energy is

$$\text{K.E.} = \tfrac{1}{2}mv^2 = \underline{\underline{\tfrac{1}{2}m\omega^2a^2\sin^2\omega t + \tfrac{1}{2}mu^2\cos^2\omega t}}$$

Add these to get the total energy, the result is

$$\underline{\underline{E = K.E. + P.E. = \tfrac{1}{2}m\omega^2a^2 + \tfrac{1}{2}mu^2}}$$

The total energy doesn't depend on time, even though the kinetic and potential energies are both time-dependent. This is because total energy is conserved.

Calculate the angular momentum vector from $\mathbf{L} = m\mathbf{r}\times\mathbf{v}$.

$$\mathbf{L} = m\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\omega t & \frac{u}{\omega}\sin\omega t & 0 \\ -a\omega\sin\omega t & u\cos\omega t & 0 \end{vmatrix} = m\mathbf{k}\left(au\cos^2\omega t + au\sin^2\omega t\right)$$

$$\Rightarrow \quad \underline{\underline{\mathbf{L} = (0,0,mau)}}$$

$\mathbf{L}$ is independent of t because angular momentum is conserved .

[11 marks]

If you want to make the particle's path a circle, what value should you give to the initial velocity u?

For a circular path we need to have

$$\mathbf{r} = (a\cos\omega t, a\sin\omega t, 0) \qquad \text{or} \quad \mathbf{r} = (a\cos\omega t, -a\sin\omega t, 0)$$

which are circular orbits with the radius a. To do that we must have

$$\underline{\underline{u = \pm a\omega\ .}}$$

[2 marks]

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3. *A particle of mass m moving in a central potential $V(r)$ has two conserved quantities*

$$\begin{aligned} J &= mr^2\frac{d\theta}{dt} \, , \\ \text{and} \quad E &= \tfrac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{2mr^2} + V(r) \, . \end{aligned}$$

Here r and θ are the position of the particle in polar coordinates.

Comment briefly on the physical interpretation of J and E.

[**4 marks**]

J is the angular momentum.

E is the total energy.

Use the chain rule to prove

$$\frac{dr}{dt} = \frac{J}{mr^2}\left(\frac{dr}{d\theta}\right)$$

and hence show that

$$\frac{J^2}{2mr^4}\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\} = E - V(r) \, . \qquad (1)$$

[**10 marks**]

This can be done several ways. For example, start with the chain rule result

$$\frac{dr}{dt} = \frac{d\theta}{dt}\frac{dr}{d\theta}$$

Try to remove $\frac{d\theta}{dt}$, because it isn't in the result we were asked to prove. We can do this from our definition of J,

$$J = mr^2\frac{d\theta}{dt} \quad \Rightarrow \quad \frac{d\theta}{dt} = \frac{J}{mr^2}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{J}{mr^2}\left(\frac{dr}{d\theta}\right)$$

as required.

Substitute this into the energy equation

$$E = \tfrac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{2mr^2} + V(r) = \frac{m}{2}\left(\frac{J}{mr^2}\frac{dr}{d\theta}\right)^2 + \frac{J^2}{2mr^2} + V(r)$$

$$\Rightarrow \qquad \underline{\underline{\frac{J^2}{2mr^4}\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\} = E - V(r)}}$$

which is the result we were asked to prove.

A particle moving in a central potential follows the path

$$r = \frac{B}{\sqrt{\sin 2\theta}} \qquad \textit{for } 0 < \theta < \frac{\pi}{2}\,.$$

Use the result (1) to find the potential $V(r)$ in terms of r and the constants in the problem, E, J and B.

[11 marks]

$$r = B\,(\sin 2\theta)^{-\frac{1}{2}} \quad \Rightarrow \quad \frac{dr}{d\theta} = -\,\tfrac{1}{2}B\,(\sin 2\theta)^{-\frac{3}{2}}\,2\,\cos 2\theta$$

Substitute this into the result we proved in the previous part of this question

$$\begin{aligned} E - V(r) &= \frac{J^2}{2m}\,\frac{\sin^2 2\theta}{B^4}\left\{\frac{B^2\cos^2 2\theta}{\sin^3 2\theta} + \frac{B^2}{\sin 2\theta}\right\} \\ &= \frac{J^2}{2m}\,\frac{\sin^2 2\theta}{B^4}\,\frac{B^2(\cos^2 2\theta + \sin^2 2\theta)}{\sin^3 2\theta} \\ &= \frac{J^2}{2mB^2}\,\frac{1}{\sin 2\theta}\,. \end{aligned}$$

This isn't quite the answer yet, because we are asked for the answer in terms of r, not θ. Use

$$r = B\,(\sin 2\theta)^{-\frac{1}{2}} \quad \Rightarrow \quad \sin 2\theta = \frac{B^2}{r^2}$$

to eliminate θ, giving the final answer

$$\underline{\underline{V(r) = E - \frac{J^2 r^2}{2mB^4}}}\,.$$

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4. (a) *Find the moment of inertia of a thin uniform square slab of mass M with sides of length $2a$, about an axis through its centre, and normal to its plane.*

We will need the density, ρ, of the lamina later on. The total area of the square is

$$A = (2a)^2 = 4a^2$$

so the density is

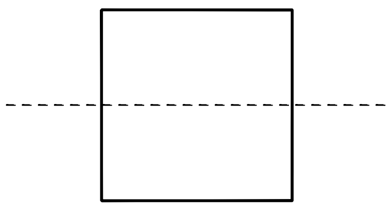
$$\rho = \frac{M}{A} = \frac{M}{4a^2}\ .$$

Now work out the moment of inertia

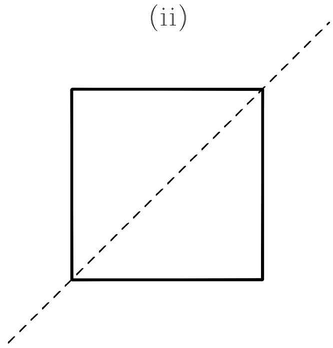
$$\begin{aligned}
I_{zz} &= \int dm(x^2+y^2) = \int \rho\, dA\ (x^2+y^2) \\
&= \rho \int_{-a}^{a} dx \int_{-a}^{a} dy (x^2+y^2) \\
&= \rho \int_{-a}^{a} x^2\, dx \int_{-a}^{a} dy + \rho \int_{-a}^{a} dx \int_{-a}^{a} y^2\, dy \\
&= \rho \left[\tfrac{1}{3}x^3\right]_{-a}^{a} [y]_{-a}^{a} + \rho\, [x]_{-a}^{a} \left[\tfrac{1}{3}y^3\right]_{-a}^{a} \\
&= \rho\, \tfrac{2}{3}a^3\ 2a + \rho\, 2a\ \tfrac{2}{3}a^3 = \tfrac{8}{3}\rho a^4 \\
\Rightarrow \qquad & \underline{\underline{I_{zz} = \tfrac{2}{3}Ma^2}}
\end{aligned}$$

State the Perpendicular Axes theorem for a lamina and hence or otherwise find the moment of inertia of the slab if it is rotating around an axis in the plane of the lamina as shown in figure (i) and when rotating about the axis shown in figure (ii).

(i)



(ii)



The Perpendicular Axes theorem states that the moment of inertia about an axis normal to the lamina is the sum of the moments of inertia about any two perpendicular axes in the plane of the lamina, if all three of these axes intersect at the same point.

So for diagram (i) the theorem says that

$$I_{zz} = I_{xx} + I_{yy}$$

By symmetry $I_{xx} = I_{yy}$ so

$$\underline{\underline{I_{xx} = \tfrac{1}{2} I_{zz} = \tfrac{1}{3} M a^2}}$$

Likewise the two diagonals of the square must have equal moments of inertia by symmetry, so

$$\underline{\underline{I_{\text{diag}} = \tfrac{1}{2} I_{zz} = \tfrac{1}{3} M a^2}}$$

(i.e. the moments of inertia are the same for diagrams (i) and (ii))

[**9 marks**]

(b) *Explain briefly the meaning of the terms* **principal axes** *and* **principal moments of inertia**. *Describe a set of principal axes for the square lamina above. What are the principal moments of inertia about these axes?*

The principal axes are the axes which make the moment of inertia tensor diagonal, i.e. they are the eigenvectors of I.

The principal moments of inertia are the diagonal entries in I when the principal axes are used, i.e. the eigenvalues of I.

For the square lamina in this question the principal axes are the normal to the lamina, and any two perpendicular axes in the plane, with all three axes passing through the midpoint of the square.

The principal moments are

$$\underline{\underline{\tfrac{2}{3} M a^2, \quad \tfrac{1}{3} M a^2, \quad \tfrac{1}{3} M a^2 \ .}}$$

[**6 marks**]

(c) *A ring and a coin, both of mass M and radius R, are released from rest at the top of a ramp of height h with a slope of $45°$, and roll down to the base. Use energy conservation to calculate their final speeds.*

Which will reach the bottom first?

Calculate the time each object will need to reach the base of the ramp.

[You may use the results

$$\begin{aligned} I &= \tfrac{1}{2} M R^2 && \textit{for a coin,} \\ I &= M R^2 && \textit{for a ring.]} \end{aligned}$$

The angular velocity ω of a rolling object is

$$\omega = \frac{v}{R}$$

so it has total kinetic energy (from centre-of-mass motion and rotation combined) of

$$T = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\frac{v^2}{R^2}$$

For the coin (putting in the value of I from the hint)

$$T = \tfrac{1}{2}Mv^2 + \tfrac{1}{4}MR^2\frac{v^2}{R^2} = \underline{\underline{\tfrac{3}{4}Mv^2}}$$

For the ring

$$T = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}MR^2\frac{v^2}{R^2} = \underline{\underline{Mv^2}}$$

Both objects start with potential energy Mgh at the top of the ramp, which is completely converted to kinetic when the object gets to the bottom of the ramp.
So for the coin energy conservation gives

$$\tfrac{3}{4}Mv^2 = Mgh \quad \Rightarrow \quad \underline{\underline{v = \sqrt{\tfrac{4}{3}gh}}} \qquad \text{coin}$$

and for the ring

$$Mv^2 = Mgh \quad \Rightarrow \quad \underline{\underline{v = \sqrt{gh}}} \qquad \text{ring}$$

The coin rolls faster, so it will reach the bottom first.
To find the time, use the formula for uniform acceleration

$$\text{distance} = \tfrac{1}{2}(v_{init} + v_{final})t\ .$$

In both cases the initial velocity $v_{init} = 0$, we have just found the final velocities v_{final}. Since the ramp is at $45°$ the distance the objects roll is $\sqrt{2}\,h$, so

$$\sqrt{2}\,h = \tfrac{1}{2}v_{final}t \quad \Rightarrow \quad t = \frac{h\sqrt{8}}{v_{final}}$$

For the coin

$$t = \frac{h\sqrt{8}}{\sqrt{\tfrac{4}{3}gh}} = \underline{\underline{\sqrt{\frac{6h}{g}}}} \qquad \text{coin}$$

and for the ring

$$t = \frac{h\sqrt{8}}{\sqrt{gh}} = \underline{\underline{\sqrt{\frac{8h}{g}}}} \qquad \text{ring}$$

[**10 marks**]

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5. (a) *Explain briefly what is meant by* **an inertial frame of reference.**

An inertial frame of reference is a frame in which particles with no external forces acting on them move in straight lines with uniform velocity.

[2 marks]

An inertial frame S' is moving with velocity v along the x-axis relative to another frame S. The Lorentz transformation converting the coordinates in S to coordinates in S' is

$$t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z$$

$$\text{where} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \, .$$

If two events happen in frame S at the same time, what are the conditions that they will also be simultaneous in the frame S' ?

Let event a have the coordinates (t, x_a, y_a, z_a) and event b have the coordinates (t, x_b, y_b, z_b) in the frame S (they happen at the same time, t, but different positions).

Using the Lorentz transformation we find their times in the frame s' will be

$$\underline{\underline{t'_a = \gamma\left(t - \frac{v}{c^2}x_a\right)}} \qquad \text{and} \qquad \underline{\underline{t'_b = \gamma\left(t - \frac{v}{c^2}x_b\right)}}$$

The events are simultaneous in S' if $t'_a = t'_b$, which implies

$$\underline{\underline{x_a = x_b}}$$

(the y and z coordinates don't matter).

[4 marks]

In frame S a particle moves with the three-velocity $\mathbf{u} = (u_x, u_y, u_z)$. If it is at the origin at $t = 0$, calculate its coordinates in frame S at time t.

Transform these coordinates into frame S'. Use your result to calculate the particle's three-velocity $\mathbf{u}'$ in frame S'.

If the particle starts at the origin and moves with constant velocity $\mathbf{u}$ its coordinate in the initial frame are simply

$$(t, x, y, z) = (t, \; u_x t, \; u_y t, \; u_z t).$$

Transforming this into the S' frame gives

$$\begin{aligned}
t' &= \gamma\left(t - \frac{v}{c^2}u_x t\right) = \gamma\left(1 - \frac{vu_x}{c^2}\right)t \\
x' &= \gamma(u_x t - vt) = \gamma(u_x - v)t \\
y' &= u_y t \\
z' &= u_z t
\end{aligned}$$

To find the velocity in the new frame we need the position as a function of the transformed time t', not the original time t. To eliminate t use

$$t = \frac{t'}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

to give

$$\begin{aligned}
x' &= \frac{(u_x - v)t'}{1 - \frac{vu_x}{c^2}} \\
y' &= \frac{u_y t'}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \\
z' &= \frac{u_z t'}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}.
\end{aligned}$$

Now we can read off the velocity in the new frame

$$\begin{aligned}
u'_x &= \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \\
u'_y &= \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \\
u'_z &= \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}.
\end{aligned}$$

[**9 marks**]

(b) *A stationary particle of mass M decays into a particle with rest mass m and a photon with rest mass zero, conserving total energy and momentum. Find the energy and momentum carried by the photon.*

Initially the total energy of the system is the rest mass of the particle which decays, $E = Mc^2$. The total momentum is 0. Let E_γ and $\mathbf{p}_\gamma$ be the energy and momentum of the photon, and E_m and $\mathbf{p_m}$ the energy and momentum of the other particle.

Conservation of energy and momentum tell us that

$$\begin{aligned} E_m + E_\gamma &= Mc^2 & (1) \\ \mathbf{p_m} + \mathbf{p}_\gamma &= \mathbf{0} \quad \Rightarrow \quad \mathbf{p_m}^2 = \mathbf{p}_\gamma^2 & (2) \end{aligned}$$

We also know that the 4-momentum of each particle can be used to calculate the rest mass

$$\begin{aligned} P_m \cdot P_m \equiv E_m^2 - \mathbf{p_m}^2 c^2 &= (mc^2)^2 & (3) \\ P_\gamma \cdot P_\gamma \equiv E_\gamma^2 - \mathbf{p}_\gamma^2 c^2 &= 0 & (4) \end{aligned}$$

(because the photon's rest mass is 0).
We now have to solve these four equations to find E_γ in terms of the masses before and after the decay. This can be done in lots of ways. One good way is to start by taking the difference between (3) and (4)

$$\begin{aligned} & E_m^2 - E_\gamma^2 - \mathbf{p_m}^2 c^2 + \mathbf{p}_\gamma^2 c^2 = m^2 c^4 & \quad (3) - (4) \\ \Rightarrow \quad & E_m^2 - E_\gamma^2 = m^2 c^4 & \quad \text{using } (2) \end{aligned}$$

Next use $E_m = Mc^2 - E_\gamma$ (equation (1)) to eliminate E_m.

$$\begin{aligned} & E_m^2 - E_\gamma^2 = m^2 c^4 \\ \Rightarrow \quad & (Mc^2 - E_\gamma)^2 - E_\gamma^2 = m^2 c^4 \\ \Rightarrow \quad & M^2 c^4 - 2Mc^2 E_\gamma = m^2 c^4 \\ \Rightarrow \quad & E_\gamma = \frac{(M^2 - m^2)c^2}{2M} \end{aligned}$$

$$p_\gamma = E_\gamma / c = \frac{(M^2 - m^2)c}{2M}$$

[**10 marks**]

A popular (wrong) answer was

$$E_\gamma = Mc^2 - mc^2 \ ,$$

which is the answer you get if you ignore the kinetic energy of the particle m, which is a reasonable approximation if $E_\gamma \ll Mc^2$. However, to get full marks in a relativity question you should calculate the exact answer, taking kinetic energy into account, as done here.