PAPER CODE NO.
MATH285

THE UNIVERSITY
of LIVERPOOL

JANUARY 2005 EXAMINATIONS

Bachelor of Science : Year 1
Bachelor of Science : Year 2
Master of Physics : Year 2

MECHANICS AND RELATIVITY

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES: Attempt FOUR questions only.

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1. (a) A particle of mass m moving along the x-axis is acted on by a force

$$F(t) = A\cos\alpha t - Bt^2$$

at time t, where A, α and B are constants.

Using Newton's law, calculate the velocity $\dot{x}(t)$ and the position $x(t)$ given that $\dot{x} = 0$ and $x = 0$ when $t = 0$.

[10 marks]

(b) A particle with mass m is moving on the x-axis under the influence of a force given by

$$F(x) = A - Kx$$

where A and K are constants. Calculate the potential energy function $V(x)$ for this force, assuming that $V(x) = 0$ at $x = 0$.

Hence write down the particle's total energy if it is at position x and has velocity v.

[7 marks]

Take the mass of the particle to be 1 kg, and the values of the constants to be $A = 2$ N and $K = 2$ N m^{-1}. The particle is launched from $x = 0$ with an initial velocity of 2 m s^{-1}.

What is its total energy?

Express the velocity v as a function of x. Use this to calculate the positions of both turning points of the particle.

[8 marks]

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2. A particle of mass m moves in three dimensions in the potential

$$V(x, y, z) = \tfrac{1}{2} m\omega^2 (x^2 + y^2 + z^2),$$

where ω is a constant.

Calculate the force acting on the particle. Use this to show that the equations of motion for the particle are

$$\begin{aligned} \ddot{x} &= -\omega^2 x \\ \ddot{y} &= -\omega^2 y \\ \ddot{z} &= -\omega^2 z \,. \end{aligned}$$

Find the general solutions of the equations for x, y and z.

[12 marks]

At $t = 0$ the particle's position and velocity are

$$\begin{aligned} \mathbf{r} &= (a, 0, 0) \\ \dot{\mathbf{r}} &= (0, u, 0) \,. \end{aligned}$$

Find $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$.

Use this solution to calculate the potential energy, the kinetic energy and the angular momentum at time t. Comment on the physical reason for your answers.

[11 marks]

If you want to make the particle's path a circle, what value should you give to the initial velocity u?

[2 marks]

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3. A particle of mass m moving in a central potential $V(r)$ has two conserved quantities

$$\begin{aligned} J &= mr^2 \frac{d\theta}{dt} \,, \\ \text{and} \quad E &= \tfrac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{J^2}{2mr^2} + V(r) \,. \end{aligned}$$

Here r and θ are the position of the particle in polar coordinates.

Comment briefly on the physical interpretation of J and E.

[**4 marks**]

Use the chain rule to prove

$$\frac{dr}{dt} = \frac{J}{mr^2} \left(\frac{dr}{d\theta}\right)$$

and hence show that

$$\frac{J^2}{2mr^4} \left\{ \left(\frac{dr}{d\theta}\right)^2 + r^2 \right\} = E - V(r) \,. \tag{1}$$

[**10 marks**]

A particle moving in a central potential follows the path

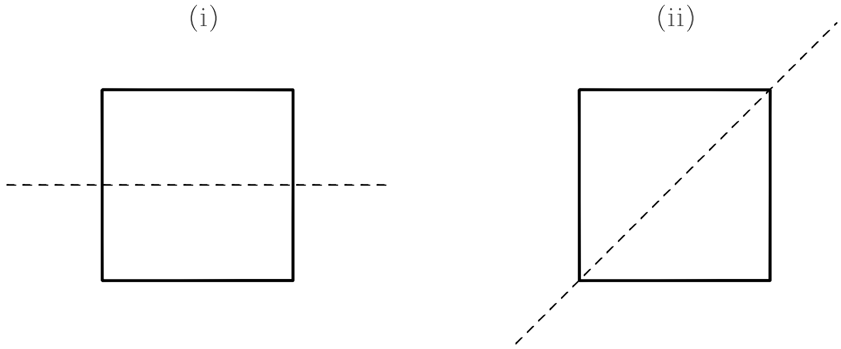
$$r = \frac{B}{\sqrt{\sin 2\theta}} \qquad \text{for } 0 < \theta < \frac{\pi}{2} \,.$$

Use the result (1) to find the potential $V(r)$ in terms of r and the constants in the problem, E, J and B.

[**11 marks**]

4. (a) Find the moment of inertia of a thin uniform square slab of mass M with sides of length $2a$, about an axis through its centre, and normal to its plane.

State the Perpendicular Axes theorem for a lamina and hence or otherwise find the moment of inertia of the slab if it is rotating around an axis in the plane of the lamina as shown in figure (i) and when rotating about the axis shown in figure (ii).



[9 marks]

(b) Explain briefly the meaning of the terms *principal axes* and *principal moments of inertia*. Describe a set of principal axes for the square lamina above. What are the principal moments of inertia about these axes?

[6 marks]

(c) A ring and a coin, both of mass M and radius R, are released from rest at the top of a ramp of height h with a slope of $45°$, and roll down to the base. Use energy conservation to calculate their final speeds.

Which will reach the bottom first?

Calculate the time each object will need to reach the base of the ramp.

[You may use the results

$$I = \tfrac{1}{2}MR^2 \qquad \text{for a coin,}$$
$$I = MR^2 \qquad \text{for a ring.]}$$

[10 marks]

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5. (a) Explain briefly what is meant by *an inertial frame of reference.*

[2 marks]

An inertial frame S' is moving with velocity v along the x-axis relative to another frame S. The Lorentz transformation converting the coordinates in S to coordinates in S' is

$$t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z$$

$$\text{where} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \ .$$

If two events happen in frame S at the same time, what are the conditions that they will also be simultaneous in the frame S' ?

[4 marks]

In frame S a particle moves with the three-velocity $\mathbf{u} = (u_x, u_y, u_z)$. If it is at the origin at $t = 0$, calculate its coordinates in frame S at time t.

Transform these coordinates into frame S'. Use your result to calculate the particle's three-velocity $\mathbf{u}'$ in frame S'.

[9 marks]

(b) A stationary particle of mass M decays into a particle with rest mass m and a photon with rest mass zero, conserving total energy and momentum. Find the energy and momentum carried by the photon.

[10 marks]