THE UNIVERSITY
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Math 284

May 2005 Examination

"Linear Algebra for Year 2 Physicists"

Full marks will be awarded for complete answers to FOUR questions. Only the best 4 answers will be taken into account. Note that each question carries a total of 20 marks that are distributed as stated.

`This is a half unit course.`
`As agreed, the exam counts for 90% while homework for 10%.`

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1. **[20 marks]**

(a) Explain how to determine the dimension of the subspace $V = \text{span}(v_1, v_2, \ldots, v_r)$ with $r \leq n$ and all $v_j \in R^n$.

[2 marks]

(b) From the following three matrices [2 marks]

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ -3 & 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

identify the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

(c) Show that $\lambda = 5$ is an eigenvalue of the matrix

$$B = \begin{bmatrix} 2 & 1 & 3 & 0 & 5 \\ 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 \\ 9 & 9 & 0 & 5 & 9 \\ 9 & 0 & 0 & 0 & 6 \end{bmatrix}.$$

[4 marks]

(d) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}.$$

[7 marks]

Further solve the system of differential equations

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

[5 marks]

2. **[20 marks]**

Consider the initial value problem

$$\frac{dy}{dx} = \sin(x + y - 2), \qquad y(0) = 3.$$

Find approximations to $y(0.2)$

(a) using the explicit Euler method with a steplength $h = 0.1$; [8 marks]

(b) using the Runge-Kutta method with a steplength $h = 0.2$; [9 marks]

$$\begin{aligned} w_0 &= y_0 = c, \\ w_{n+1} &= w_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ \text{with} \quad k_1 &= f(x_n, w_n), \\ k_2 &= f(x_n + \tfrac{1}{2}h, w_n + \tfrac{1}{2}hk_1), \\ k_3 &= f(x_n + \tfrac{1}{2}h, w_n + \tfrac{1}{2}hk_2), \\ k_4 &= f(x_n + h, w_n + hk_3). \end{aligned}$$

Given that the exact solution is $y^* = y(0.2) = 3.1844$, which of the above two methods is more accurate? [3 marks]

3. **[20 marks]**

(1) In each of the following cases, determine (giving your reason) whether the given vectors are linearly dependent or linearly independent:

(1a) $z_1 = (3 \;\; -1 \;\; 2)$, $\quad z_2 = (-9 \;\; 3 \;\; -6)$;

(1b) $m_1 = (-3 \;\; 2 \;\; 0)$, $\quad m_2 = (-4 \;\; 3 \;\; 1)$, $\quad m_3 = (1 \;\; 1 \;\; 1)$;

(1c) $v_1 = (0 \;\; 0 \;\; 1 \;\; 3)$, $\quad v_2 = (5 \;\; 3 \;\; 5 \;\; 5)$, $\quad v_3 = (0 \;\; 0 \;\; 0 \;\; 2)$, $\quad v_4 = (0 \;\; 9 \;\; 5 \;\; 4)$.

[6 marks]

Further determine the dimension of the following subspaces

(1a) $\text{span}\,(z_1,\ z_2)$;

(1b) $\text{span}\,(m_1,\ m_2,\ m_3)$;

(1c) $\text{span}\,(v_1,\ v_2,\ v_3,\ v_4)$.

[3 marks]

(2) Find the rank and the nullity of the matrix

$$A = \begin{bmatrix} -1 & -1 & -1 & -2 \\ 1 & 2 & 3 & 3 \\ 1 & 3 & 5 & 4 \end{bmatrix}.$$

[5 marks]

Further verify that the following two vectors belong to the null space of A

$$n_1 = (0 \;\; -3 \;\; 1 \;\; 1)^T, \quad n_2 = (1 \;\; 1 \;\; 0 \;\; -1)^T.$$

[4 marks]

Could the two vectors n_1, n_2 form a basis for the null space of A (give your reasons)?

[2 marks]

4. **[20 marks]**

Show that the eigenvalues λ of the matrix

$$A = \begin{bmatrix} 3 & 4 & 3 \\ 4 & 3 & 3 \\ 3 & 3 & 4 \end{bmatrix}$$

satisfy the cubic equation

$$\lambda^3 - 10\lambda^2 - \lambda + 10 = 0.$$

[5 marks]

Furthermore,

(a) given that one eigenvalue is $\lambda = 10$, find the other eigenvalues. [5 marks]

(b) verify that a set of eigenvectors corresponding to the eigenvalues is

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

[4 marks]

(c) find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$. [2 marks]

(d) identify the form of the quadric defined by

$$3x^2 + 3y^2 + 4z^2 + 8xy + 6xz + 6yz = 1.$$

[4 marks]

5. **[20 marks]**

(a) Let the subspace $V = \text{span}(v_1, v_2)$ in R^7 with

$$v_1 = (2 \;\; 0 \;\; 0 \;\; 5 \;\; 0 \;\; 5 \;\; -1), \qquad v_2 = (0 \;\; 4 \;\; \sqrt{7} \;\; 11 \;\; 0 \;\; 0 \;\; 0).$$

What is the dimension of V? [2 marks]

Find the projection decomposition $v_1 = \alpha v_2 + v_2^{\perp}$, where v_2 and $v_2^{\perp}$ are orthogonal i.e. $(v_2, v_2^{\perp}) = 0$.

[9 marks]

Assuming $\|v_2^{\perp}\| = \sqrt{4895}/12$, find an orthonormal basis for V. [3 marks]

(b) Using the relationship $\|v\|^2 = (v, v) = v^T v$, show that if $w, v \in R^n$, and $v = Pw$ with $P \in R^{n \times n}$ orthogonal, then [6 marks]

$$\|v\| = \|w\|.$$

6. **[20 marks]**

The Legendre polynomials $P_n(x)$, where $n = 0, 1, 2, ...,$ are orthogonal i.e.

$$(P_m, P_n) = \int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1} & \text{if } m = n. \end{cases}$$

The first four Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x.$$

(a) Express the cubic polynomial $13 - 3x^2 + 5x^3$ in terms of Legendre polynomials. [5 marks]

(b) Verify that the above $P_2(x)$ satisfies Legendre's equation for $n = 2$

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0.$$

[5 marks]

(c) Verify (by integration) that

$$\int_{-1}^{1} [P_3(x)]^2 dx = \frac{2}{7} \; .$$

[5 marks]

(d) Find the first four coefficients in the expansion of the even function

$$f(x) = \begin{cases} 5 + x, & -1 < x < 0 \\ 5 - x, & 0 \leq x < 1 \end{cases}$$

in Legendre polynomials. [5 marks]