PAPER CODE NO.
MATH 284

THE UNIVERSITY
of LIVERPOOL

MAY EXAMINATIONS 2007

Bachelor of Science : Year 2
Master of Physics : Year 2
Master of Physics : Year 4
No qualification aimed for : Year 1

LINEAR ALGEBRA

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Full marks will be awarded for complete answers to FOUR questions, Only the best 4 answers will be taken into account. Note that each question carries a total of 20 marks that are distributed as stated.

1. **[20 marks]**

(a) Let the subspace $V = \text{span}(v_1, v_2, \ldots, v_r)$ with all $v_j \in \mathbb{R}^n$ and the dimension of V be m. Explain the relationship of m with the integers r, n.

[2 marks]

(b) Given [6 marks]

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

find $det(A)$, matrix C of co-factors and hence the inverse of A.

(c) Find the eigenvalues and eigenvectors of the above matrix A from (b). [6 marks]
Further solve the system of differential equations

$$\frac{d\mathrm{x}}{dt} = A\mathrm{x}$$

where [6 marks]

$$\mathrm{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

2. [20 marks]

Consider the initial value problem

$$\frac{dy}{dx} = \sin(x + y - 2), \qquad y(0) = 3.$$

Find approximations to $y(0.2)$

(a) using the explicit Euler method with a steplength $h = 0.1$; [8 marks]

(b) using the Runge-Kutta method with a steplength $h = 0.2$; [9 marks]

$$\begin{aligned} w_0 &= y_0 = c, \\ w_{n+1} &= w_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ \text{with} \quad k_1 &= f(x_n, w_n), \\ k_2 &= f(x_n + \tfrac{1}{2}h, w_n + \tfrac{1}{2}hk_1), \\ k_3 &= f(x_n + \tfrac{1}{2}h, w_n + \tfrac{1}{2}hk_2), \\ k_4 &= f(x_n + h, w_n + hk_3). \end{aligned}$$

Given that the exact solution is $y^* = y(0.2) = 3.1844$, which of the above two methods is more accurate? [3 marks]

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3. **[20 marks]**

(1) In each of the following cases, determine (giving your reason) whether the given vectors are linearly dependent or linearly independent:

(1a) $z_1 = (3 \;\; -1 \;\; 2)$, $\quad z_2 = (-9 \;\; -3 \;\; -6)$;

(1b) $m_1 = (-3 \;\; 2 \;\; 1)$, $\quad m_2 = (3 \;\; -2 \;\; -1)$, $\quad m_3 = (-6 \;\; 4 \;\; 2)$;

(1c) $v_1 = (0 \;\; 0 \;\; 1 \;\; 3)$, $\quad v_2 = (2 \;\; 0 \;\; 0 \;\; 7)$, $\quad v_3 = (0 \;\; 0 \;\; 0 \;\; 2)$, $\quad v_4 = (0 \;\; 9 \;\; 5 \;\; 4)$.

[6 marks]

Further determine the dimension of the following subspaces

(1a) $\text{span}\,(z_1, \; z_2)$;

(1b) $\text{span}\,(m_1, \; m_2, \; m_3)$;

(1c) $\text{span}\,(v_1, \; v_2, \; v_3, \; v_4)$.

[3 marks]

(2) Find the rank and the nullity of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 3 & 5 & 4 \end{bmatrix}.$$

[5 marks]

Further verify that the following two vectors belong to the null space of A

$$u_1 = (1 \;\; -2 \;\; 1 \;\; 0)^T, \quad u_2 = (1 \;\; 1 \;\; 0 \;\; -1)^T.$$

[4 marks]

Could the above two vectors u_1, u_2 form a basis for the null space of A (give your reasons)?

[2 marks]

4. [20 marks]

Show that the eigenvalues λ of the matrix

$$A = \begin{bmatrix} 7 & 4 & -28 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

satisfy the cubic equation

$$\lambda^3 - 7\lambda^2 - 4\lambda + 28 = 0.$$

[5 marks]

Furthermore,

(a) given that one eigenvalue is $\lambda = 2$, find the other eigenvalues. [5 marks]

(b) verify that a set of eigenvectors corresponding to the eigenvalues is

$$\frac{1}{\sqrt{2451}}\begin{pmatrix} 49 \\ 7 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{21}}\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{21}}\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}.$$

[6 marks]

(c) find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$.
[4 marks]

5. [20 marks]

(a) Let the subspace $V = \text{span}(v_1, v_2)$ in $\mathbb{R}^8$ with

$$v_1 = (2\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1\ \ 0), \qquad v_2 = (1\ \ \sqrt{3}\ \ \sqrt{13}\ \ 0\ \ 0\ \ -1\ \ 2\ \ \sqrt{3}).$$

What is the dimension of V? [2 marks]

Find the projection decomposition $v_1 = \alpha v_2 + v_2^{\perp}$, where v_2 and $v_2^{\perp}$ are orthogonal i.e. $(v_2, v_2^{\perp}) = 0$.

[9 marks]

Assuming $\|v_2^{\perp}\| = \sqrt{134}/5$, find an orthonormal basis for V. [5 marks]

(b) Find the inverse to the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -5 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 0 & 1 \end{bmatrix}.$$

and verify your answer. [4 marks]

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6. [20 marks]

The Legendre polynomials $P_n(x)$, where $n = 0, 1, 2, \ldots,$ are orthogonal i.e.

$$(P_m, P_n) = \int_{-1}^{1} P_m(x) P_n(x) dx = \begin{cases} 0 & \text{if } m \neq n, \\ \dfrac{2}{2n+1} & \text{if } m = n. \end{cases}$$

The first four Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x.$$

(a) Express the cubic polynomial $21 - 3x^2 + 5x^3$ as a linear combination of Legendre polynomials.

[5 marks]

(b) Verify that the above $P_2(x)$ satisfies Legendre's equation for $n = 2$

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$

[5 marks]

(c) Verify (by integration) that

$$\int_{-1}^{1} [P_3(x)]^2 dx = \frac{2}{7}\ .$$

[5 marks]

(d) Find the first four coefficients in the expansion of the even function

$$f(x) = \begin{cases} 5 + x, & -1 < x < 0 \\ 5 - x, & 0 \leq x < 1 \end{cases}$$

in Legendre polynomials. [5 marks]