THE UNIVERSITY
of LIVERPOOL

Math 284
Summer 2003 Examination

"Linear Algebra for Year 2 Physicists"

Full marks will be awarded for complete answers to FOUR questions. Only the best 4 answers will be taken into account. Note that each question carries a total of 20 marks that are distributed as stated.

`This is a half unit course.`
`As agreed, the exam counts for 80% while homework for 20%.`

1. [**20 marks**]

(1) Give the definition of linear independence of r vectors $v_1, v_2, \dots, v_r$ in space R^n. [2 marks]

(2) Which of the following two matrices share the same eigenvalue $\lambda = 4$? [4 marks]

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 0 & 3 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 3 & 0 & 2 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 2 & 0 & 3 \\ 3 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, B_4 = \begin{bmatrix} 4 & 0 & 3 \\ 1 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Hint. Expanding $det(B - \lambda I)$ along a row or a column.

(3) Show that $\lambda = 7$ is an eigenvalue of the matrix [4 marks]

$$B = \begin{bmatrix} 9 & 0 & 3 & 0 & 7 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 5 & 1 & 0 & 5 & 0 \\ 7 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Find another eigenvalue of B. [2 marks]

(4) Using the matrix method, find the general solution $x_1 = x_1(t)$ and $x_2 = x_2(t)$ of the coupled equations

$$\dot{x}_1 = 4x_1 + 2x_2, \qquad \dot{x}_2 = x_1 + 3x_2.$$

[8 marks]

2. **[20 marks]**

(1) In each of the following cases, determine (giving your reason) whether the given vectors $v_1, v_2, \ldots, v_r$ are linearly dependent or linearly independent, and further determine the rank of the row space of matrix [8 marks]

$$A = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{pmatrix}$$

(1a) $r = 2$: $v_1 = (-1, -1, 4), \quad v_2 = (2, 2, -8)$;

(1b) $r = 2$: $v_1 = (1, 2, 3), \quad v_2 = (3, 2, 1)$;

(1c) $r = 3$: $v_1 = (3, 2, 1), \quad v_2 = (0, 3, 1), \quad v_3 = (0, 0, -1)$;

(1d) $r = 4$: $v_1 = (3, 2, 1, 3, 5), \quad v_2 = (0, 2, 1, 0, 4), \quad v_3 = (3, 0, 1, 0, 5)$, $v_4 = (0, 0, 1, 1, 2)$.

(2) Find the nullity of the matrix [5 marks]

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 & 5 \\ 1 & 3 & 5 & 1 & 4 \end{bmatrix}.$$

Further find a basis for the null space of A. [7 marks]

3. **[20 marks]**

Given the following 3×3 matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix},$$

(1) Show that one eigenvalue is $\lambda = 4$; [3 marks]

(2) Find the other eigenvalues. [5 marks]

(3) Find a set of all eigenvectors corresponding to the three eigenvalues. [6 marks]
Hence or otherise find an orthogonal matrix P such that $D = P^{-1}AP$ is diagonal. [3 marks]
Compute $det(A)$. [3 marks]

4. **[20 marks]**

The Legendre polynomials $P_n(x)$ where $n = 0, 1, 2, ...$ are orthogonal i.e.

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1} & \text{if } m = n. \end{cases} \tag{4.1}$$

The first few Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x.$$

(1) Verify that the above $P_3(x)$ satisfies Legendre's equation for $n = 3$ [4 marks]

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$

(2) Verify that equation (4.1) holds for the particular cases
(i) $m = 0, n = 1$; (ii) $m = n = 2$. [6 marks]

(3) Express $5x^3 - 7$ in terms of Legendre polynomials. [4 marks]

(4) Find the first two coefficients in the expansion of the following function

$$f(x) = \begin{cases} 2 + x, & -1 < x < 0 \\ x - 2, & 0 \leq x < 1 \end{cases}$$

in Legendre polynomials. [6 marks]

5. **[20 marks]**

Find approximations to $y(0.2)$ for the initial value problem

$$\frac{dy}{dx} = 3xy + 2y, \quad y(0) = \frac{1}{2},$$

(i) using the Euler method with a steplength $h = 0.1$; [9 marks]

(ii) using the Runge-Kutta method with a steplength $h = 0.2$;

$$\begin{aligned} w_0 &= y_0 = c, \\ w_{n+1} &= w_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ \text{with} \quad k_1 &= f(x_n, w_n), \\ k_2 &= f(x_n + \tfrac{1}{2}h, w_n + \tfrac{1}{2}hk_1), \\ k_3 &= f(x_n + \tfrac{1}{2}h, w_n + \tfrac{1}{2}hk_2), \\ k_4 &= f(x_n + h, w_n + hk_3). \end{aligned}$$

[8 marks]

Given that the exact solution is $y = y(x) = \frac{1}{2}\exp\left(\frac{1}{2}x(3x+4)\right)$, find the absolute errors of the above two solutions? [2 mark]
Which method is more accurate? [1 mark]

6. **[20 marks]**

(1) Let V be the set in R^7 spanned by two vectors

$$v_1 = (2, 0, 0, 3, 0, 5, -1), \qquad v_2 = (1, 1, 1, 1, 0, 0, 1).$$

What is the orthogonal projection of v_1 in the direction of v_2? [2 marks]
Hint. $v_1 = v_2^{\perp} + proj_{v_2}(v_1) = v_2^{\perp} + \alpha v_2$.

Find an orthonormal basis u_1, u_2 for V. [6 marks]

Given a third vector $v_3 = (1, 1, 1, 1, 1, 1, 1)$ and based on the above vectors u_1, u_2, find constants β_1, β_2 and the unique vector u_3 orthogonal to u_1, u_2 such that

$$v_3 = \beta_1 u_1 + \beta_2 u_2 + u_3.$$

[5 marks]

(2) Find the inverse to the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

and verify your answer. [7 marks]