MATH284 Linear Algebra May 2001

Full marks can be obtained for FOUR complete answers. Only your best FOUR ANSWERS will count.

1. (a) Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} .$$

[6 marks]

(b) Show that $\lambda = 6$ is an eigenvalue of the matrix

$$A = \begin{pmatrix} 6 & 3 & 0 \\ 1 & 4 & 1 \\ -2 & -2 & 4 \end{pmatrix}$$

and determine the other two eigenvalues. Find eigenvectors corresponding to the eigenvalues. [13 marks]

Hence write down the general solution of the system of differential equations

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} .$$

[3 marks]

Write down a constant invertible matrix P such that if $\mathbf{x} = P\mathbf{y}$ then

$$\frac{d\mathbf{y}}{dt} = D\mathbf{y}$$

where D is a diagonal matrix which should be stated. [3 marks]

2. (a) In each of the following cases determine (giving your reasons) whether the given vectors in R^3 are linearly dependent or linearly independent:

(i) $(1,2,2)$, $(2,4,5)$;

(ii) $(1,2,1)$, $(3,1,2)$, $(-3,4,-1)$;

(iii) $(1,1,2)$, $(2,1,3)$, $(1,3,1)$.

[6 marks]

(b) Find the rank and the nullity of the matrix

$$\begin{pmatrix} 1 & 3 & 2 & 5 \\ 2 & 7 & 6 & 10 \\ 1 & 5 & 7 & 6 \\ 2 & 9 & 8 & 8 \end{pmatrix} .$$

[6 marks]

(c) Find all solutions (if there are any) of the equations

$$\begin{array}{rcrcrcrl} x_1 & + & 3x_2 & - & x_3 & - & 2x_4 & = 8 \\ 2x_1 & + & 7x_2 & + & x_3 & - & 2x_4 & = 15 \\ x_1 & + & 5x_2 & + & 6x_3 & + & 3x_4 & = 5. \end{array}$$

[7 marks]

(d) Find an orthonormal basis for the subspace of R^3 spanned by the vectors $(1,-1,1)$ and $(4,-1,1)$.

[6 marks]

3. (a) In each of the following cases determine whether or not the given matrix is orthogonal.

(i)

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(ii)

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} .$$

[4 marks]

(b) Show that the eigenvalues λ of the matrix

$$A = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

satisfy

$$\lambda^3 - 4\lambda^2 - 15\lambda + 18 = 0.$$

Given that one eigenvalue is $\lambda = 1$, find the other eigenvalues.

[6 marks]

Find normalised eigenvectors corresponding to the eigenvalues.

[9 marks]

Write down an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

[2 marks]

Identify the form of quadric surface given by

$$2y^2 + 2z^2 + 6xy + 2yz + 6xz = 10 .$$

[4 marks]

4. The solutions of Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

that are finite on the interval $[-1,1]$ are the Legendre polynomials $P_n(x)$ where $n = 0, 1, 2, \ldots$. These polynomials are orthogonal and form a basis for the set of piecewise continuous functions on the interval $[-1,1]$. They are normalised such that

$$\int_{-1}^{1} \left[P_n(x)\right]^2 dx = \frac{2}{2n+1} \; .$$

The first few Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x.$$

(i) Verify that $P_3(x)$ satisfies Legendre's equation. [3 marks]

(ii) Check that

$$\int_{-1}^{1} [P_2(x)]^2 dx = \frac{2}{5} \; .$$

[4 marks]

(iii) Express $1-x+2x^2+x^3$ in terms of Legendre polynomials. [5 marks]

(iv) Obtain a formula for the coefficients in the expansion of a function in a series of Legendre polynomials. [5 marks]

(v) Find the first four coefficients in the expansion of the function

$$f(x) = \begin{cases} -x & -1 < x < 0 \\ x & 0 \le x < 1 \end{cases}$$

in Legendre polynomials. [8 marks]

5. The Taylor series method of order r for the initial value problem

$$y' = f(x, y), \quad y(a) = c$$

is

$$w_0 = c$$

$$w_{n+1} = w_n + \sum_{s=0}^{r-1} \frac{h^{s+1}}{(s+1)!} f^{(s)}(x_n, w_n) \ .$$

Use the Taylor method with a steplength $h = 0.1$ and

(i) $r = 1$ (Euler's method),

(ii) $r = 2$

to find approximations to $y(0.2)$ for the problem

$$y' = 1 - x - x^2 + y, \quad y(0) = 1 \ .$$

[15 marks]

For the same problem and again taking a steplength $h = 0.1$ find an approximation to $y(0.2)$ using the Runge-Kutta method

$$w_0 = c$$

$$w_{n+1} = w_n + \frac{1}{2}(k_1 + k_2)$$

where

$$k_1 = hf(x_n, w_n), \quad k_2 = hf(x_{n+1}, w_n + k_1) \ .$$

[10 marks]