PAPER CODE NO.
MATH 283

THE UNIVERSITY
of LIVERPOOL

RESIT EXAMINATIONS 2007

Bachelor of Science : Year 2
Master of Physics : Year 2

FIELD THEORY AND PARTIAL DIFFERENTIAL EQUATIONS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Attempt FOUR questions only. All questions are of equal value (25 marks each). Throughout the paper $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ represent unit vectors parallel to the x, y and z axes respectively.

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1. (a) (i) Given that

$$\phi(x, y, z) = 3x^2 + y^2 + 6z^2,$$

calculate $\nabla\phi$. [2 marks]

(ii) Find the directional derivative, $\mathcal{D}_b\phi(\mathbf{a})$, of ϕ at the point $\mathbf{a} = (1, 1, 2)$ in the direction of the vector $\mathbf{b} = (1, 1, 1)$. [3 marks]

(iii) Calculate the outward unit normal to the ellipsoid

$$3x^2 + y^2 + 6z^2 = 28$$

at the point $\mathbf{a} = (1, 1, 2)$. [4 marks]

(iv) Hence find the Cartesian equation of the tangent plane to the ellipsoid at the point $\mathbf{a} = (1, 1, 2)$ [4 marks]

(b) (i) Using the definition of gradient (∇) and curl ($\nabla\times$), show that

$$\nabla \times (\phi\mathbf{v}) = \phi\nabla \times \mathbf{v} + \nabla\phi \times \mathbf{v}$$

for any (smooth enough) scalar field ϕ and vector field $\mathbf{v}$.

[4 marks]

(ii) Verify that

$$\nabla \times (\mathbf{r}) = \mathbf{0}\ ,\ \nabla\left(r^2\right) = 2\mathbf{r}\ ,\ \text{and}\ \nabla\left(\frac{1}{r^2}\right) = -\frac{2}{r^4}\mathbf{r}\ ,\ r \neq 0\ ,$$

where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $r = |\mathbf{r}|$. [4 marks]

(iii) Using the results of parts (b)(i) and (ii), deduce the expression for

$$\nabla \times \nabla\left(r^2\right)\ ,\ r \neq 0\ .$$

Briefly discuss the result. [4 marks]

2. (a) Evaluate the line integral

$$\int_{C_1} \mathbf{F} \cdot \mathbf{dr}$$

where

$$\mathbf{F} = (2x + y)\hat{\mathbf{i}} + (x + 2y)\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$$

and the curve C_1 is the helix parametrized by the equations

$$x(t) = \cos t\ ,\ y(t) = \sin t\ , z(t) = t\ ,\ 0 \leq t \leq 2\pi\ .$$

[7 marks]

(b) Using the definition of gradient (∇) and curl ($\nabla\times$), show that

$$\nabla \times \nabla\phi = \mathbf{0}\ ,$$

for any (smooth enough) scalar function ϕ.

[4 marks]

(c) Show that the vector field $\mathbf{F}$ in (a) is irrotational, that is, $\nabla \times \mathbf{F} = \mathbf{0}$.

[5 marks]

Deduce that $\mathbf{F}$ can be expressed as the gradient of a scalar field, ϕ, and find this scalar field.

[5 marks]

(d) Evaluate the line integral

$$\int_{C_2} \mathbf{F} \cdot \mathbf{dr}\ ,$$

where C_2 is the straight line segment from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$. Compare with the result of (a) and comment briefly.

[4 marks]

3. (a) State Gauss's Theorem for a differentiable vector field $\mathbf{F}$ defined over a volume τ with bounding surface S, clearly explaining the meaning of the terms in the theorem. [10 marks]

(b) The region τ is the ellipsoid enclosed by the surface $x^2 + 2y^2 + 3z^2 = 4$. Use Gauss's Theorem to evaluate

$$\int\int_S (2x\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\ dS,$$

where S is the bounding surface and $\hat{\mathbf{n}}$ the outward unit normal to S. [15 marks]

4. (a) State Stokes's Theorem for a differentiable vector field $\mathbf{F}$ defined over a surface S bounded by a closed curve C. [5 marks]

(b) Calculate the curl of the vector field

$$\mathbf{F} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}.$$

Hence determine whether $\mathbf{F}$ is a conservative field. [5 marks]

(c) Evaluate, by direct integration, the integral

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} dS$$

where S is the disc

$$x^2 + y^2 \leq a^2$$

in the $z = 2$ plane, oriented by the upward unit normal $\hat{\mathbf{k}}$. [7 marks]

Verify your result using Stokes's Theorem. (Hint: you may use the parameters $x = a\cos t, y = a\sin t, z = 2$, where $0 \leq t \leq 2\pi$). [8 marks]

5. A scalar function $V(x, y)$ obeys Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{1}$$

in the square region $(0 \le x \le 2\pi), (0 \le y \le 2\pi)$, and is subject to the following boundary conditions

$$V(x,0) = 0,\ V(x,2\pi) = 0, \quad 0 \le x \le 2\pi, \tag{2}$$
$$V(0,y) = 0,\ V(2\pi,y) = V_0, \quad 0 \le y \le 2\pi, \tag{3}$$

where V_0 denotes a constant.

(a) Use separation of variables $V(x,y) = X(x)Y(y)$ to show that (1) decouples into

$$\frac{d^2X}{dx^2} - \alpha^2 X = 0,\ \alpha \ne 0 \tag{4}$$

and

$$\frac{d^2Y}{dy^2} + \alpha^2 Y = 0,\ \alpha \ne 0. \tag{5}$$

[6 marks]

(b) From (2) and (3), deduce the boundary conditions associated with (4) and (5).

Hence show that the eigenvalues of (4) and (5) are

$$\alpha_n = \frac{n}{2}\ ,\ n = 1,\ 2, \ldots$$

and their associated eigenfunctions are

$$X_n(x) = D_n \sinh\left(\frac{nx}{2}\right), \qquad Y_n(y) = B_n \sin\left(\frac{ny}{2}\right).$$

[8 marks]

(c) Show that the solution of (1) - (3) can be expressed as

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1}^{\infty} \frac{\sinh\left((2n-1)\frac{x}{2}\right)}{(2n-1)\sinh((2n-1)\pi)} \sin\left((2n-1)\frac{y}{2}\right).$$

[Hint: you may assume that $\int_0^{2\pi} \sin\left(\frac{ny}{2}\right) \sin\left(\frac{ky}{2}\right) dy = \pi$, if $n = k$, $n \ne 0$, and 0 otherwise.] [11 marks]

6. The displacement $V(x,t)$ from the horizontal of a uniform elastic string of unstretched length a satisfies the wave equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}\,, \tag{6}$$

where c is a strictly positive constant (the speed of the wave).
This equation is subject to the boundary conditions

$$V(0,t) = V(a,t) = 0 \tag{7}$$

together with the initial condition

$$V(x,0) = 0, \qquad 0 < x < a \tag{8}$$

(a) Show, using separation of variables $V(x,t) = X(x)T(t)$, that equation (6) decouples into

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0,\ \ \alpha \neq 0 \tag{9}$$

and

$$\frac{d^2T}{dt^2} + c^2\alpha^2 T = 0,\ \ \alpha \neq 0. \tag{10}$$

[6 marks]

(b) Deduce that the most general solution of (6) - (8) is

$$V(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi ct}{a}\right).$$

[5 marks]

(c) Find the constants E_n given that

$$\frac{\partial V}{\partial t}(x,0) = \pi,\ \ 0 \le x \le a. \tag{11}$$

[7 marks]

Hence show that the solution of (6) - (8) and (11) is

$$V(x,t) \;=\; \frac{4a}{\pi c}\left[\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi x}{a}\right)\sin\left(\frac{(2n-1)\pi ct}{a}\right)\right].$$

[Hint: you may assume that $\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{k\pi x}{a}\right)dx = \frac{a}{2}$ if $n = k, n \neq 0$, and 0 otherwise.] [7 marks]