PAPER CODE NO.
MATH283

THE UNIVERSITY
of LIVERPOOL

ANWERS TO SEPTEMBER 2006 EXAMINATIONS

Bachelor of Engineering: Year 2
Bachelor of Science: Year 2
Master of Engineering: Year 2
Master of Physics: Year 2

FIELD THEORY AND PARTIAL DIFFERENTIAL EQUATIONS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Attempt FOUR questions only. All questions are of equal value (25 marks each).

In this paper $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ represent unit vectors parallel to the x, y and z axes respectively and $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

1. (a) Given that

$$\phi(x, y, z) = 2x^2 + y^2 + z^2 \quad ,$$

find $\nabla\phi$. Deduce the magnitude and the greatest rate of change of Φ at point $(1, 1, 0)$.

Further, calculate the outward pointing unit normal to the ellipsoid

$$2x^2 + y^2 + z^2 = 3$$

at the point $(1, 1, 0)$. Use this to find the cartesian equation of the tangent plane at this point.

[15 marks]

(b) Calculate the divergence of the vector function

$$\mathbf{v} = \frac{1}{r^2}\,\mathbf{r}\ ,\ \mathbf{r} \neq \mathbf{0}\ ,$$

where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $r = |\mathbf{r}|$.

[10 marks]

Answers:

(a) The gradient of Φ is

$$\nabla\phi = (4x, 2y, 2z)\ .$$

[2 marks]

Hence, the magnitude of the greatest rate of ϕ at the point $(1, 1, 0)$ is

$$|\ \nabla\phi\ | = \sqrt{4^2 + 2^2 + 0} = 2\sqrt{5}\ .$$

[2 marks]

The unit *outward* pointing normal to the ellipsoid

$$2x^2 + y^2 + z^2 = 3$$

at a general point (x, y, z) is (note that the sign is positive)

$$\hat{\mathbf{n}} = \frac{\nabla\Phi}{|\ \nabla\Phi\ |} = \frac{(4x, 2y, 2z)}{\sqrt{16x^2 + 4y^2 + 4z^2}} = \frac{(2x, y, z)}{\sqrt{4x^2 + y^2 + z^2}}\ .$$

[2 marks]

At point $\mathbf{a} = (1, 1, 0)$ it can be recast as:

$$\hat{\mathbf{n}} = \frac{(2, 1, 0)}{\sqrt{5}}\ .$$

[2 marks]

Hence, the cartesian equation for the tangent plane which touches the ellipsoid at that point is given by

$$\begin{aligned}
&(\mathbf{r}-\mathbf{a})\cdot\hat{\mathbf{n}} = 0\\
&(\mathbf{r}-(1,1,0))\cdot\frac{(2,1,0)}{\sqrt{5}} = 0\\
&2(x-1)+(y-1) = 0
\end{aligned}$$

Or

$$2x + y = 3\ .$$

[7 marks]

(b) To calculate the divergence of the vector function

$$\mathbf{v} = \frac{1}{r^2}\mathbf{r}\quad ,$$

where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, and $r = |\mathbf{r}|$, we write

$$\begin{aligned}
\nabla\cdot(r^{-2}\mathbf{r}) &= (\partial/\partial x, \partial/\partial y, \partial/\partial z)\cdot\frac{(x,y,z)}{r^2}\\
&= \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^2}\right)\\
&= x\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right) + \frac{1}{r^2}\frac{\partial x}{\partial x} + y\frac{\partial}{\partial y}\left(\frac{1}{r^2}\right) + \frac{1}{r^2}\frac{\partial y}{\partial y} + z\frac{\partial}{\partial z}\left(\frac{1}{r^2}\right) + \frac{1}{r^2}\frac{\partial z}{\partial z}
\end{aligned}$$

[5 marks]

We then note that

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right) &= -2r^{-3}\frac{\partial r}{\partial x}\\
&= -2r^{-3}\frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{1/2}\right)\\
&= -2r^{-3}\frac{x}{r} = -2r^{-4}x\ .
\end{aligned}$$

[3 marks]

A similar expression holds for the other two variables y and z. We thus find that

$$\begin{aligned}
\nabla\cdot(r^{-2}\mathbf{r}) &= x^2\left(-\frac{2}{r^4}\right) + \frac{1}{r^2} + y^2\left(-\frac{2}{r^4}\right) + \frac{1}{r^2} + z^2\left(-\frac{2}{r^4}\right) + \frac{1}{r^2}\\
&= \frac{3}{r^2} - \frac{2r^2}{r^4} = \frac{1}{r^2}\ .
\end{aligned}$$

[2 marks]

2. (a) Show that for any (smooth enough) scalar field Φ

$$\nabla \times \nabla \Phi = \mathbf{0} \, .$$

Deduce that only one of the vector fields

$$\mathbf{F}_1 = (6x + 2y)\hat{\mathbf{i}} + 2x\hat{\mathbf{j}} + \hat{\mathbf{k}} \, , \qquad \mathbf{F}_2 = (2x^3 + z)\hat{\mathbf{i}} + 3xy\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}}$$

can be expressed as the gradient of a scalar field Φ. [10 marks]

(b) Evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where

$$\mathbf{F} = xy\hat{\mathbf{i}} + (x - 2y)\hat{\mathbf{j}}$$

and $\mathcal{C}$ is the curve parameterised by equations

$$\begin{aligned} x &= t \\ y &= 2t + 1 \\ z &= t^3 \end{aligned}$$

and the curve begins at $t = 0$ and ends at $t = 1$. [15 marks]

Answers:

(a) For any smooth enough scalar field ϕ

$$\begin{aligned} \nabla \times (\nabla \phi) &= \hat{\mathbf{i}} \left(\frac{\partial}{\partial y}\frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial \phi}{\partial y} \right) - \hat{\mathbf{j}} \left(\frac{\partial}{\partial x}\frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial \phi}{\partial x} \right) + \hat{\mathbf{k}} \left(\frac{\partial}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \phi}{\partial x} \right) \\ &= \hat{\mathbf{i}} \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right) - \hat{\mathbf{j}} \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right) + \hat{\mathbf{k}} \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right) \\ &= 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}} = \mathbf{0} \end{aligned} \tag{1}$$

Nota Bene: Here, we use the fact that the order upon which we take the partial derivatives does not matter (provided that the field which we consider is smooth enough by Schwarz's theorem). [5 marks]

The curl of $\mathbf{F}_1 = (6x + 2y)\hat{\mathbf{i}} + 2x\hat{\mathbf{j}} + \hat{\mathbf{k}}$ is given by

$$\nabla \times \mathbf{F}_1 = \hat{\mathbf{i}}\,(0 - 0) - \hat{\mathbf{j}}\,(0 - 0) + \hat{\mathbf{k}}\,(2 - 2) = \mathbf{0} \, .$$

We deduce that there exists a scalar field ϕ such that $\mathbf{F}_1 = \nabla \phi$. [3 marks]

The curl of $\mathbf{F}_2 = (2x^3 + z)\hat{\mathbf{i}} + 3xy\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}}$ writes as

$$\nabla \times \mathbf{F}_2 = \hat{\mathbf{i}}\,(0-0) - \hat{\mathbf{j}}\left(z^2 - 1\right) + \hat{\mathbf{k}}\,(3y - 0) \neq \mathbf{0}\ .$$

Hence, it does not derive from a scalar field. [2 marks]

Nota Bene:

$$\mathbf{F}_1 = (6x + 2y)\hat{\mathbf{i}} + 2x\hat{\mathbf{j}} + \hat{\mathbf{k}} = \nabla\phi\ .$$

In other words,

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= 6x + 2y \\ \frac{\partial\phi}{\partial y} &= 2x \\ \frac{\partial\phi}{\partial z} &= 1\ , \end{aligned}$$

so that

$$\begin{aligned} \phi &= 3x^2 + 2xy + f_1(x,y) \\ \phi &= 2xy + f_2(x,z) \\ \phi &= z + f_3(x,y)\ , \end{aligned}$$

where f_1, f_2 and f_3 are three arbitrary functions which we need specify. By inspection, we end up with

$$\phi = 3x^2 + 2xy + z + C$$

where C is an arbitrary constant.

(b) First, we note that

$$\mathbf{F}(t) = t(2t+1)\hat{\mathbf{i}} + (t - 4t - 2)\hat{\mathbf{j}}\ ,$$

and

$$\mathbf{r}(t) = t\hat{\mathbf{i}} + (2t+1)\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}\ .$$

We thus have

$$\frac{d\mathbf{r}}{dt} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}\ .$$

[8 marks]

The line integral is therefore

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \left((2t^2 + t)\hat{\mathbf{i}} - (3t+2)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}})\right) dt \\ &= \int_0^1 \left(2t^2 + t - 6t - 4\right) dt \\ &= \left[2\frac{t^3}{3} - 5\frac{t^2}{2} - 4t\right]_0^1 = \frac{2}{3} - \frac{5}{2} - 4 = -\frac{35}{6}\ . \end{aligned}$$

[7 marks]

Alternative derivation:

One can also write

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_{\mathcal{C}} \Big(xy\hat{\mathbf{i}} + (x-2y)\hat{\mathbf{j}} \Big) \cdot \Big(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} \Big) \\
&= \int_{\mathcal{C}} xy\, dx + \int_{\mathcal{C}} (x-2y)\, dy \\
&= \int_0^1 (2t^2 + t)\, dt + \int_0^1 (t - 4t - 2)\, 2dt \\
&= \left[2\frac{t^3}{3} + \frac{t^2}{2} \right]_0^1 + 2\left[-\frac{3t^2}{2} - 2t \right]_0^1 \\
&= \frac{2}{3} + \frac{1}{2} - 3 - 4 = -\frac{35}{6}\ .
\end{aligned}$$

3. State Gauss's theorem for a differentiable vector field $\mathbf{F}$ defined over a volume τ with bounding surface S.

[10 marks]

Let S be the surface of the region τ bounded by the planes $x =$, $y = 0$, $z = 0$, $z = 3$ and $x + 2y = 6$. Sketch the region τ and use Gauss's theorem to evaluate

$$\iint_S (2xz\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + y^2 z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS \quad ,$$

where S is the bounding surface and $\hat{\mathbf{n}}$ the outward unit normal to S.

[15 marks]

Answers:

Let us state Gauss's theorem for a differentiable vector field $\mathbf{F}$ defined over a volume τ with bounding surface S.

Gauss's (or divergence's) theorem :

Given a volume τ which is bounded by a piecewise continuous surface S, and a vector function $\mathbf{F}$ which is continuous and has continuous partial derivatives on a region which includes $\tau \cup S$, then [5 marks]

$$\iiint_\tau \nabla \cdot \mathbf{F} d\tau = \oint_S \mathbf{F} \cdot d\mathbf{S} \quad . \tag{2}$$

[5 marks]

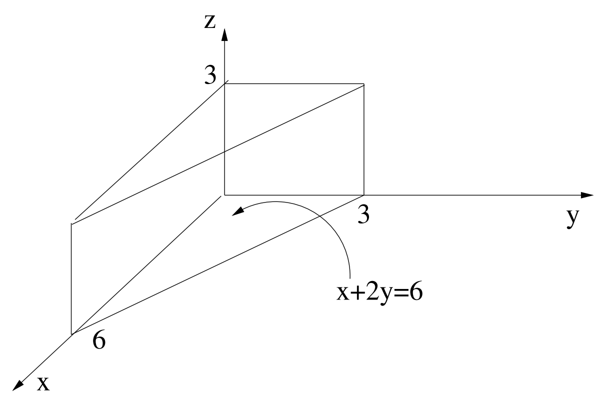
Notice that the surface integral S has been drawn with a circle around it, in order to indicate that this surface is *closed*, i.e. that it entirely encompasses the volume τ. Also, $d\mathbf{S}$ is defined as the product of a small area (let's say $dxdy$) by the unit outward normal $\hat{\mathbf{n}}$ to the surface S.

Using Gauss's theorem, the surface integral of the solid depicted on Figure can be expressed as

$$\begin{aligned}
& \iint_S (2xz\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + y^2 z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS \\
:= \; & \oint_S \left[2xz\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + y^2 z\hat{\mathbf{k}}\right] \cdot d\mathbf{S} \\
= \; & \iiint_\tau \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left[2xz\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + y^2 z\hat{\mathbf{k}}\right] d\tau \\
= \; & \int_0^3 \int_0^3 \int_0^{6-2y} (y^2 + 2z + x)\, dxdydz
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^3\int_0^3 \left[y^2x + 2zx + \frac{x^2}{2}\right]_0^{6-2y} dydz \\
&= \int_0^3 \left[(y^2z + z^2)(6-2y) + \frac{z}{2}(6-2y)^2\right]_0^3 dy \\
&= \int_0^3 \left[18y^2 - 6y^3 + 54 - 18y + \frac{3}{2}(6-2y)^2\right]_0^3 dy \\
&= \left[6y^3 - \frac{3}{2}y^4 + 54y - 9y^2 + \frac{1}{2}(6-2y)^3\left(-\frac{1}{2}\right)\right]_0^3 \\
&= (6\times 3^3 - \frac{3}{2}\times 3^4 - 54\times 3 - 9\times 3^2) - \frac{1}{2}6^3\left(-\frac{1}{2}\right) \\
&= 3^4\left(\frac{3}{2}\right) + 3^2\times 6 = \frac{248}{2} + \frac{108}{2} = \frac{351}{2}\ . \qquad (3)
\end{aligned}
$$

[15 marks]



4. State Stokes' theorem for a differentiable vector field $\mathbf{F}$ over a surface S bounded by a closed curve $\mathcal{C}$.

[10 marks]

Calculate the curl of the vector field

$$\mathbf{F} = (x^3 - 3y^3)\hat{\mathbf{i}} + xy^2\hat{\mathbf{j}} + xyz\hat{\mathbf{k}} \ .$$

Hence determine whether or not $\mathbf{F}$ is a conservative field.

[5 marks]

Use Stokes' theorem to evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \ ,$$

where C is the closed curve in the plane $z = 0$ formed by the x-axis, the line $x = 3$ and the curve $y = x^3$.

Briefly discuss the result. [10 marks]

Answers:

Let us first state the Stokes' theorem for a differentiable vector field $\mathbf{F}$ defined over a surface S bounded by a closed curve $\mathcal{C}$.
Stokes' theorem:
Given a surface S which is bounded by a piecewise continuous curve $\mathcal{C}$, and a vector function $\mathbf{F}$ which is continuous and has continuous partial derivatives on a region which includes $S \cup \mathcal{C}$, then [5 marks]

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \tag{4}$$

[5 marks]

The curl of the vector field

$$\mathbf{F} = (x^3 - 3y^3)\hat{\mathbf{i}} + xy^2\hat{\mathbf{j}} + xyz\hat{\mathbf{k}} \ ,$$

writes as

$$\begin{aligned} \nabla \times \mathbf{F} &= (xz - 0)\,\hat{\mathbf{i}} - (yz - 0)\,\hat{\mathbf{j}} + \hat{\mathbf{k}}\left(y^2 + 9y^2\right) \\ &= xz\hat{\mathbf{i}} - yz\hat{\mathbf{j}} + 10y^2\hat{\mathbf{k}} \ . \end{aligned}$$

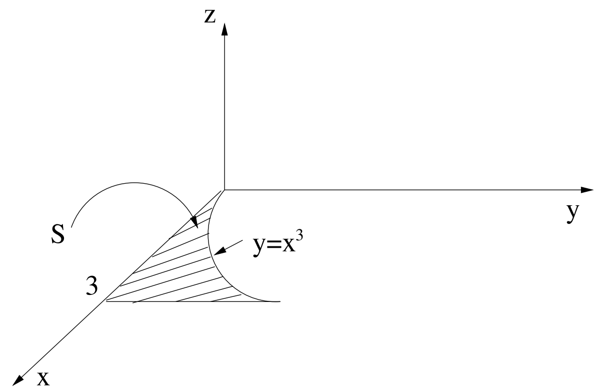
[3 marks]

The curl of the vector field $\mathbf{F}$ is different from the null vector, therefore $\mathbf{F}$ is not a conservative field. [2 marks]

Bu using Stokes' theorem the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ can be evaluated thanks to the surface integral

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \ ,$$

where S is the plane surface bounded by the closed curve C defined in the plane $z = 0$, formed by the x-axis, the line $x = 3$ and the curve $y = x^3$. It is depicted on the figure below so that the unit normal $\hat{\mathbf{n}}$ to



the surface $\mathbf{S}$ is simply the vector of the canonical basis $\hat{\mathbf{k}}$ (we deduce the orientation of $\hat{\mathbf{n}}$ from the thumb rule). Thus, an infinitesimal element $d\mathbf{S}$ of orientable surface $\mathbf{S}$ writes as $d\mathbf{S} = \hat{\mathbf{n}}dxdy = \hat{\mathbf{k}}dxdy$ and we end up with

$$\begin{aligned}
\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int\int_S (10y^2\hat{\mathbf{k}}) \cdot \hat{\mathbf{k}}dxdy = \int_0^3\int_0^{x^3} 10y^2 dxdy \\
&= \int_0^3 \left[\frac{10y^3}{3}\right]_0^{x^3} dx = \frac{10}{3}\int_0^3 x^9 dx \\
&= \left[\frac{x^{10}}{3}\right]_0^3 = 3^9 = 1962 \ .
\end{aligned}$$

[8 marks]

We have just shown that the line integral of $\mathbf{F}$ over the closed curve $\mathcal{C}$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 1962 \ ,$$

is not null. The reason for that is that $\mathbf{F}$ is not conservative. [2 marks]

5. A scalar function $V(x, y)$ obeys Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{5}$$

in the rectangular region $(0 \le x \le \pi), (0 \le y \le \pi)$, and is subject to the following boundary conditions

$$V(x, 0) = x \;, V(x, \pi) = 0 \;, \tag{6}$$
$$\frac{\partial V}{\partial x}(0, y) = \frac{\partial V}{\partial x}(\pi, y) = 0 \;. \tag{7}$$

(a) Use separation of variables $V(x, y) = X(x)Y(y)$ to show that (5) decouples into

$$\frac{d^2 X}{dx^2} + \alpha^2 X = 0 \;, \; \alpha \neq 0 \;, \tag{8}$$

and

$$\frac{d^2 Y}{dy^2} - \alpha^2 Y = 0 \;, \; \alpha \neq 0 \;. \tag{9}$$

[6 marks]

From (6) and (7), deduce the boundary conditions associated with (8) and (9). Hence show that the eigenvalues of (8) and (9) are

$$\alpha = n \;, \; n = 0, \; 1, \; 2, \cdots$$

and their associated eigenvectors are

$$X_n(x) = A_n \cos(nx) \;, Y_n(y) = C_n \frac{\sinh(n(\pi - y))}{\sinh(n\pi)} \;.$$

[10 marks]

(b) Finally, show that the solution of the boundary value problem (5)-(7) can be expressed as

$$V(x, y) = \frac{\pi - y}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \frac{\sinh(n(\pi - y))}{\sinh(n\pi)} \cos(nx) \;.$$

[Hint: you may assume that $\int_0^\pi \cos(ny)\cos(ky)dy = \frac{\pi}{2}$, if $n = k$, $n \neq 0$. If $n = k = 0$, the integral is π. The integral is 0 otherwise.]

[9 marks]

Answers:

(a) We try a solution which has the form

$$V(x,y) = X(x)Y(y) \quad , \tag{10}$$

and now we have to work out what these functions $X(x)$ and $Y(y)$ are. Substituting into Laplace's equation, we obtain

$$X''(x)Y(y) + Y''(y)X(x) = 0 \tag{11}$$

[2 marks]

which we can rearrange to get

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \quad . \tag{12}$$

[1 marks]

So the variables separate, and we see that we have to solve the two differential equations

$$\frac{d^2X}{dx^2} = \pm\alpha^2 X(x) \quad , \frac{d^2Y}{dy^2} = \mp\alpha^2 Y(y) \quad . \tag{13}$$

[1 marks]

To get the right choice for the sign in front of α^2 we look at the boundary conditions: The boundary conditions $(\partial/\partial x)V(0,y) = 0$ and $(\partial/\partial x)V(\pi,y) = 0$, $0 \le y \le \pi$, imply that

$$X'(0) = X'(\pi) = 0 \quad , \tag{14}$$

[1 marks]

and so the function $X(x)$ cannot end up being made up of cosh and sinh functions otherwise it would be identically zero, and so the differential equations which we have to solve are

$$\frac{d^2X}{dx^2} = -\alpha^2 X(x) \quad , \tag{15}$$

$$\frac{d^2Y}{dy^2} = +\alpha^2 Y(y) \quad . \tag{16}$$

[1 marks]

The general solution to the first differential equation (15) is

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x) \; . \tag{17}$$

[1 marks]

Hence,

$$X'(x) = -\alpha A \sin(\alpha x) + \alpha B \cos(\alpha x) \ . \tag{18}$$

[1 marks]

The boundary condition $X'(0) = 0$ allows us to eliminate one of the constants:

$$0 = -\alpha A \sin 0 + \alpha B \cos 0 \ ,$$

hence $\alpha = 0$ or $B = 0$. [1 marks]

Let us assume for now that $\alpha \neq 0$. Applying the boundary condition $X'(\pi) = 0$ gives us a restraint on α:

$$0 = A \sin(\alpha \pi) \ , \alpha \neq 0 \ ,$$

[1 marks]

and this implies

$$\alpha \pi = \pi, 2\pi, \ldots$$

that is,

$$\alpha = n \qquad \text{where } n = 1, 2, \ldots \tag{19}$$

[1 marks]

We thus have the eigenvalues of the differential equation (15):

$$X_n(x) = A_n \cos(nx) \quad , \text{where } n = 1, 2, \ldots \tag{20}$$

[1 marks]

We now solve the second differential equation (16). The general solution to this equation is

$$Y_n(y) = C_n \cosh(\alpha y) + D_n \sinh(\alpha y) \ . \tag{21}$$

[1 marks]

Now, looking at the boundary condition for our function $Y_n(y)$ we are provided with

$$Y_n(\pi) = C_n \cosh(n\pi) + D_n \sinh(n\pi) = 0 \ , \ n = 1, 2, ... \tag{22}$$

[1 marks]

Hence,

$$D_n = -\frac{C_n \cosh(n\pi)}{\sinh(n\pi)} \ , \ n = 1, 2, ...$$

[1 marks]

which leads to

$$\begin{aligned} Y_n(y) &= C_n \frac{(\cosh(n\pi y)\sinh(n\pi) - \cosh(n\pi)\sinh(n\pi y))}{\sinh(n\pi)} \\ &= C_n \frac{(\sinh(n(\pi - y))}{\sinh(n\pi)} \ . \end{aligned}$$

[1 marks]

Let us now assume that $\alpha = n = 0$. In this case, (16) implies

$$\frac{d^2Y}{dy^2} = 0 \ ,$$

[1 marks]

so that $Y = a + by$ (it is a linear function). Since $Y(\pi) = 0$, and using the principle of superposition, which says that we can add any two solutions together to get another solution, we can write the general solution as

$$V(x,y) = \sum_{n=1}^{\infty} A_n C_n \frac{\sinh(n(\pi - y))}{\sinh(n\pi)} \cos(nx) + A_0 C_0 (\pi - y) \ . \tag{23}$$

[1 marks]

Now if we pick the coefficients $A_n C_n$ correctly then we can satisfy the final boundary condition, which is

$$V(x,0) = x \ , \ 0 \le x \le \pi \ .$$

This means that we want to pick coefficients C_n such that

$$x = \sum_{n=0}^{\infty} A_n C_n \cos(nx) \ , \tag{24}$$

[1 marks]

so that

$$\pi A_0 C_0 = \frac{1}{\pi} \int_0^{\pi} x dx = \frac{1}{2\pi} \ .$$

Hence, $A_0 C_0 = 1/2$. [1 marks]

In order to work out the rest of the unknown coefficients $A_n C_n$ we have to exploit the orthogonality of the functions $\cos(nx)$, that is:

$$\int_0^{\pi} \cos(nx)\cos(kx) dx = \begin{cases} \frac{\pi}{2} & \text{if } n = k, \ n \neq 0, \\ \pi & \text{if } n = k = 0, \\ 0 & \text{otherwise} \ . \end{cases} \tag{25}$$

We now multiply equation (24) by the function $\cos(kx)$, and integrate it between 0 and π:

$$\begin{aligned} \int_0^\pi x\cos(kx)dx &= \sum_{n=1}^{\infty} C_n \int_0^\pi \cos(nx)\cos(kx)dx \\ &= C_k\frac{\pi}{2}\ , \end{aligned} \tag{26}$$

[1 marks]

where we have used the orthogonality relation (25) to eliminate all terms but one from the infinite sum.

Integrating by parts, we obtain:

$$\begin{aligned} A_k C_k\,\frac{\pi}{2} &= \left[\frac{\sin(kx)}{k}x\right]_0^\pi - \int_0^\pi \left[\frac{\sin(kx)}{k}\right] dx \\ &= \left[\frac{\cos(kx)}{k^2}\right]_0^\pi \\ &= \frac{\cos(k\pi)-1}{k^2} = \frac{(-1)^k-1}{k^2}\ . \end{aligned} \tag{27}$$

[2 marks]

We can now write down the full solution to the differential equation:

$$\begin{aligned} V(x,y) &= A_0C_0(\pi-y) + \sum_{n=1}^{\infty} A_nC_n\frac{\sinh(n(\pi-y))}{\sinh(n\pi)}\cos(nx) \\ &= \frac{\pi-y}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}\frac{\sinh(n(\pi-y))}{\sinh(n\pi)}\cos(nx)\ . \end{aligned} \tag{28}$$

[2 marks]

6. The displacement $V(x,t)$ from the horizontal of a uniform elastic string of unstretched length a satisfies the wave equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}\ , \tag{29}$$

where c is a strictly positive constant (speed of wave).
This equation is subject to the boundary conditions

$$V(0,t) = V(a,t) = 0\ . \tag{30}$$

(a) Show, using separation of variables $V(x,t) = X(x)T(t)$, that (29) decouples into

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0\ ,\ \alpha \neq 0\ , \tag{31}$$

and

$$\frac{d^2T}{dt^2} + c^2\alpha^2 T = 0\ ,\ \alpha \neq 0\ . \tag{32}$$

[6 marks]

Deduce that the most general solution of (29)-(30) is

$$V(x,t) = \sum_{n=1}^{\infty} B_n \sin(\alpha_n x)\left[C_n \cos(\alpha_n ct) + D_n \sin(\alpha_n ct)\right]\ ,\ \alpha_n = \frac{\pi}{a}n\ .$$

[5 marks]

(b) Find the constants B_nC_n and C_nD_n given the initial conditions

$$V(x,0) = f(x)\ ,\ \frac{\partial V}{\partial t}(x,0) = 0\ , 0 \leq x \leq a\ . \tag{33}$$

[7 marks]

Show that the solution of the boundary value problem (29)-(30) and (33) can be expressed as

$$V(x,t) = \frac{2}{a}\sum_{n=1}^{\infty}\Big(\int_0^a f(x)\sin\frac{n\pi x}{a}dx\Big)\sin\frac{n\pi x}{a}\sin\frac{n\pi ct}{a}\ . \tag{34}$$

[Hint: you may assume that $\int_0^a \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi m x}{a}\right)dx = \frac{a}{2}$, if $n = m$, $n \neq 0$, and 0 otherwise.] [7 marks]

Answers:

(a) We try a solution which has the form

$$V(x,t) = X(x)T(t) \quad , \tag{35}$$

and now we have to work out what these functions $X(x)$ and $T(t)$ are. Substituting into the wave equation, we obtain

$$X''(x)T(t) = \frac{1}{c^2}T''(t)X(x) \tag{36}$$

which we can rearrange to get

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} \ . \tag{37}$$

[2 marks]

So once again the variables separate, and we see that we have to solve the two differential equations

$$\frac{d^2X}{dx^2} = \pm\alpha^2 X(x) \quad , \frac{1}{c^2}\frac{d^2T}{dt^2} = \pm\alpha^2 T(t) \quad . \tag{38}$$

[2 marks]

To get the right choice for the sign of α^2 we look at the boundary conditions: the fact that for all t, $V = 0$ at $x = 0$ and $x = a$, implies that

$$X(0) = X(a) = 0 \quad , \tag{39}$$

and so this function cannot end up being made up of cosh and sinh functions otherwise it would be zero, and so the differential equations we have to solve are

$$\frac{d^2X}{dx^2} = -\alpha^2 X(x) \quad , \tag{40}$$

$$\frac{d^2T}{dt^2} = -c^2\alpha^2 T(t) \quad . \tag{41}$$

[2 marks]

The general solution to the first differential equation (40) is

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x) \ . \tag{42}$$

[1 marks]

The boundary condition $X(0) = 0$ lets us get rid of one of the arbitrary constants:

$$0 = A\cos 0 + B\sin 0 \ ,$$

and so $A = 0$. Applying the boundary condition $X(a) = 0$ gives us a restraint on α:

$$0 = B\sin(\alpha a)\ ,$$

and this implies

$$\alpha a = 0, \pi, 2\pi, \ldots$$

that is,

$$\alpha = \frac{n\pi}{a} \quad \text{where } n = 0, 1, 2, \ldots \tag{43}$$

[1 marks]

We now have the eigenvalues of the differential equation (40):

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{a}\right) \quad \text{, where } n = 0, 1, 2, \ldots \tag{44}$$

[1 marks]

We now turn attempt to solve the second differential equation (41). The general solution to this equation is

$$T(t) = C\cos(\alpha ct) + D\sin(\alpha ct) \tag{45}$$

From (43), we obtain:

$$T_n(t) = C_n \cos(\frac{n\pi ct}{a}) + D_n \sin(\frac{n\pi ct}{a}) \tag{46}$$

Combining the functions of x and t, we obtain

$$\begin{aligned} V_n(x,t) &= X_n(x)T_n(t) \\ &= B_n \sin\left(\frac{n\pi x}{a}\right)\left[C_n \cos(\frac{n\pi ct}{a}) + D_n \sin(\frac{n\pi ct}{a})\right] \end{aligned}$$

[1 marks]

We now use the principle of superposition, which says that we can add any two solutions together to get another solution, to write the general solution

$$V(x,t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\left[C_n \cos(\frac{n\pi ct}{a}) + D_n \sin(\frac{n\pi ct}{a})\right]\ . \tag{47}$$

[1 marks]

(b) To fix the constants, we should now use the initial boundary condition $\frac{\partial V}{\partial t}(x,0) = 0$ (the velocity of the string is initially zero). For this let us first evalutate

$$\frac{\partial V}{\partial t}(x,t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\frac{n\pi c}{a}\left[-C_n \sin(\frac{n\pi ct}{a}) + D_n \cos(\frac{n\pi ct}{a})\right]\ , \tag{48}$$

[1 marks]

so that at $t = 0$, for every $x \in [0, \pi]$,

$$0 = \frac{\partial V}{\partial t}(x, 0) = \sum_{n=0}^{\infty} B_n D_n \sin\left(\frac{n\pi x}{a}\right) \frac{n\pi c}{a} \ . \tag{49}$$

This implies that $B_n D_n = 0$. [2 marks]

Further, the displacement of the string at $t = 0$ is

$$f(x) = V(x, 0) = \sum_{n=0}^{\infty} B_n C_n \sin\left(\frac{n\pi x}{a}\right) \ . \tag{50}$$

[2 marks]

Clearly the coefficient $B_0 C_0$ can be anything at all, since $\sin(0) = 0$, hence we set $B_0 C_0 = 0$. [1 marks]

In order to work out the rest of the unknown coefficients $B_n C_n$ we have to exploit the orthogonality of the functions $\sin \frac{n\pi x}{a}$. This means that the functions are linearly independent, that is, is is impossible to make up, say, the function $\sin \frac{3\pi x}{a}$ out of any linear combination of $\sin \frac{\pi x}{a}$, $\sin \frac{2\pi x}{a}$, $\sin \frac{4\pi x}{a}$, etc. More practically it means that multiples of orthogonal functions always integrate to zero over certain regions. In this case

$$\int_0^a \sin \frac{n\pi x}{a} \sin \frac{k\pi x}{a} \, dx = \begin{cases} \frac{a}{2} & \text{if } n = k,\ n \neq 0, \\ 0 & \text{otherwise} \end{cases} \tag{51}$$

We now multiply equation (50) by the function $\sin \frac{k\pi x}{a}$, and integrate it between 0 and a:

$$\begin{aligned} \int_0^a f(x) \sin \frac{k\pi x}{a} dx &= \sum_{n=1}^{\infty} B_n C_n \int_0^a \sin \frac{n\pi x}{a} \sin \frac{k\pi x}{a} \, dx \\ &= B_k C_k \frac{a}{2} \end{aligned} \tag{52}$$

where we have used the orthogonality relation (51) to eliminate all terms but one from the infinite sum. [4 marks]

We can now write down the full solution to the differential equation:

$$\begin{aligned} V(x, t) &= \sum_{n=1}^{\infty} B_n C_n \sin \frac{n\pi x}{a} \sin \frac{n\pi ct}{a} \\ &= \frac{2}{a} \sum_{n=1}^{\infty} \Big(\int_0^a f(x) \sin \frac{n\pi x}{a} dx \Big) \sin \frac{n\pi x}{a} \sin \frac{n\pi ct}{a} \ . \end{aligned} \tag{53}$$

[3 marks]