PAPER CODE NO.
MATH283

THE UNIVERSITY
of LIVERPOOL

SEPTEMBER 2006 EXAMINATIONS

Bachelor of Engineering: Year 2
Bachelor of Science: Year 2
Master of Engineering: Year 2
Master of Physics: Year 2

FIELD THEORY AND PARTIAL DIFFERENTIAL EQUATIONS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Attempt FOUR questions only. All questions are of equal value (25 marks each).

In this paper $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ represent unit vectors parallel to the x, y and z axes respectively, and $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

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1. (a) Given that

$$\phi(x,y,z) = 2x^2 + y^2 + z^2 \quad ,$$

find $\nabla\phi$. Deduce the magnitude and the direction of the greatest rate of change of ϕ at the point $(1,1,0)$.

Further, calculate the outward pointing unit normal to the ellipsoid

$$2x^2 + y^2 + z^2 = 3 \ ,$$

at the point $(1,1,0)$. Use this to find the cartesian equation of the tangent plane at this point.

[15 marks]

(b) Calculate the divergence of the vector function

$$\mathbf{v} = \frac{1}{r^2}\mathbf{r} \quad , \ \mathbf{r} \neq \mathbf{0}$$

where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $r = |\mathbf{r}|$.

[10 marks]

2. (a) Show that for any (smooth enough) scalar field Φ

$$\nabla \times \nabla\Phi = \mathbf{0} \ .$$

Deduce that only one of the vector fields

$$\mathbf{F}_1 = (6x + 2y)\hat{\mathbf{i}} + 2x\hat{\mathbf{j}} + \hat{\mathbf{k}} \ , \qquad \mathbf{F}_2 = (2x^3 + z)\hat{\mathbf{i}} + 3xy\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}}$$

can be expressed as the gradient of a scalar field Φ. [10 marks]

(b) Evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where

$$\mathbf{F} = xy\hat{\mathbf{i}} + (x - 2y)\hat{\mathbf{j}}$$

and $\mathcal{C}$ is the curve parameterised by equations

$$\begin{aligned} x &= t \\ y &= 2t + 1 \\ z &= t^3 \end{aligned}$$

and the curve begins at $t = 0$ and ends at $t = 1$. [15 marks]

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3. State Gauss's theorem for a differentiable vector field $\mathbf{F}$ defined over a volume τ with bounding surface S.

[10 marks]

Let S be the surface of the region τ bounded by the planes $x =$, $y = 0$, $z = 0$, $z = 3$ and $x + 2y = 6$. Sketch the region τ and use Gauss's theorem to evaluate

$$\int\int_S (2xz\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + y^2z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS \quad ,$$

where S is the bounding surface and $\hat{\mathbf{n}}$ the outward unit normal to S.

[15 marks]

4. State Stokes' theorem for a differentiable vector field $\mathbf{F}$ over a surface S bounded by a closed curve $\mathcal{C}$.

[10 marks]

Calculate the curl of the vector field

$$\mathbf{F} = (x^3 - 3y^3)\hat{\mathbf{i}} + xy^2\hat{\mathbf{j}} + xyz\hat{\mathbf{k}} \ .$$

Hence determine whether or not $\mathbf{F}$ is a conservative field.

[5 marks]

Use Stokes' theorem to evaluate the line integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \ ,$$

where C is the closed curve in the plane $z = 0$ formed by the x-axis, the line $x = 3$ and the curve $y = x^3$.

Briefly discuss the result. [10 marks]

5. A scalar function $V(x,y)$ obeys Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{1}$$

in the rectangular region $(0 \le x \le \pi), (0 \le y \le \pi)$, and is subject to the following boundary conditions

$$V(x,0) = x \ , V(x,\pi) = 0 \ , \tag{2}$$

$$\frac{\partial V}{\partial x}(0,y) = \frac{\partial V}{\partial x}(\pi,y) = 0 \ . \tag{3}$$

(a) Use separation of variables $V(x,y) = X(x)Y(y)$ to show that (1) decouples into

$$\frac{d^2 X}{dx^2} + \alpha^2 X = 0 \ , \ \alpha \neq 0 \ , \tag{4}$$

and

$$\frac{d^2 Y}{dy^2} - \alpha^2 Y = 0 \ , \ \alpha \neq 0 \ . \tag{5}$$

[6 marks]

From (2) and (3), deduce the boundary conditions associated with (4) and (5). Hence show that the eigenvalues of (4) and (5) are

$$\alpha = n \ , \ n = 0, \ 1, \ 2, \cdots$$

and their associated eigenvectors are

$$X_n(x) = A_n \cos(nx) \ , Y_n(y) = C_n \frac{\sinh(n(\pi - y))}{\sinh(n\pi)} \ .$$

[10 marks]

(b) Finally, show that the solution of the boundary value problem (1)-(3) can be expressed as

$$V(x,y) = \frac{\pi - y}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \frac{\sinh(n(\pi - y))}{\sinh(n\pi)} \cos(nx) \ .$$

[Hint: you may assume that $\int_0^\pi \cos(ny)\cos(ky)dy = \frac{\pi}{2}$, if $n = k$, $n \neq 0$. If $n = k = 0$, the integral is π. The integral is 0 otherwise.]

[9 marks]

6. The displacement $V(x,t)$ from the horizontal of a uniform elastic string of unstretched length a satisfies the wave equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}\ , \tag{6}$$

where c is a strictly positive constant (speed of wave).
This equation is subject to the boundary conditions

$$V(0,t) = V(a,t) = 0\ . \tag{7}$$

(a) Show, using separation of variables $V(x,t) = X(x)T(t)$, that (6) decouples into

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0\ ,\ \alpha \neq 0\ , \tag{8}$$

and

$$\frac{d^2T}{dt^2} + c^2\alpha^2 T = 0\ ,\ \alpha \neq 0\ . \tag{9}$$

[6 marks]

Deduce that the most general solution of (6)-(7) is

$$V(x,t) = \sum_{n=1}^{\infty} B_n \sin(\alpha_n x)\left[C_n \cos(\alpha_n ct) + D_n \sin(\alpha_n ct)\right]\ ,\ \alpha_n = \frac{\pi}{a}n\ .$$

[5 marks]

(b) Find the constants B_nC_n and C_nD_n given the initial conditions

$$V(x,0) = f(x)\ ,\ \frac{\partial V}{\partial t}(x,0) = 0\ , 0 \leq x \leq a\ . \tag{10}$$

[7 marks]

Show that the solution of the boundary value problem (6)-(7) and (10) can be expressed as

$$V(x,t) = \frac{2}{a}\sum_{n=1}^{\infty}\Big(\int_0^a f(x)\sin\frac{n\pi x}{a}dx\Big)\sin\frac{n\pi x}{a}\sin\frac{n\pi ct}{a}\ . \tag{11}$$

[Hint: you may assume that $\int_0^a \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi m x}{a}\right)dx = \frac{a}{2}$, if $n = m$, $n \neq 0$, and 0 otherwise.] [7 marks]