PAPER CODE NO.
MATH283

THE UNIVERSITY
of LIVERPOOL

SEPTEMBER 2005 EXAMINATIONS

Bachelor of Engineering: Year 2
Bachelor of Science: Year 2
Master of Engineering: Year 2
Master of Physics: Year 2

FIELD THEORY AND PARTIAL DIFFERENTIAL EQUATIONS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Attempt FOUR questions only. All questions are of equal value (25 marks each).

In this paper $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ represent unit vectors parallel to the x, y and z axes respectively, and $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

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1. (a) Given that

$$\phi(x, y, z) = x^2 + y^2 + 3z^2 \quad ,$$

find $\nabla\phi$. Hence calculate the outward pointing unit normal to the ellipsoid

$$x^2 + y^2 + 3z^2 = 5$$

at the point $(1, 1, 1)$. Use this to find the cartesian equation of the tangent plane at this point.

[15 marks]

(b) Calculate the divergence of the vector function

$$\mathbf{v} = \frac{1}{r}\,\mathbf{r} \quad , \ \mathbf{r} \neq \mathbf{0}$$

where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $r = |\mathbf{r}|$.

[10 marks]

2. (a) Evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr} \quad ,$$

where

$$\mathbf{F} = xy\hat{\mathbf{i}} + (x - y)\hat{\mathbf{j}}$$

and the curve $\mathcal{C}$ is the triangle with vertices $(0, 1)$, $(1, -1)$ and $(-1, -1)$, traversed in a clockwise direction.

[15 marks]

(b) Show that the vector field

$$\mathbf{F} = (y + z)\hat{\mathbf{i}} + (2z + x)\hat{\mathbf{j}} + (x + 2y)\hat{\mathbf{k}} \ ,$$

can be expressed as the gradient of a scalar field, and find that scalar field.

[10 marks]

3. State Gauss's theorem for a differentiable vector field $\mathbf{F}$ defined over a volume τ with bounding surface S.

[10 marks]

The region τ is the sphere enclosed by the surface $x^2 + y^2 + z^2 = 48$. Use Gauss's theorem to evaluate

$$\int\int_S (-x\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS$$

where S is the bounding surface and $\hat{\mathbf{n}}$ the outward unit normal to S.

[15 marks]

4. State Stokes' theorem for a differentiable vector field $\mathbf{F}$ over a surface S bounded by a closed curve $\mathcal{C}$.

[10 marks]

Calculate the curl of the vector field

$$\mathbf{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}} \,.$$

Hence determine whether or not $\mathbf{F}$ is a conservative field.

[5 marks]

Evaluate by direct integration the integral

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS \,,$$

where the surface S is the disk

$$x^2 + y^2 \leq a^2 \,,$$

in the $z = 0$ plane, oriented by the upward unit normal $\hat{\mathbf{n}}$.

[4 marks]

Verify your result using Stokes' theorem. (Hint: you may use the parameterisation $x = a\cos t$, $y = a\sin t$, $z = 0$, where $0 \leq t < 2\pi$).

[6 marks]

5. A scalar function $V(x,y)$ satisfies Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

in the region $(0 \le x \le \pi)$, $(0 \le y \le a)$, together with the boundary conditions

$$\begin{aligned} V(0,y) &= 0, \quad 0 \le y \le a, \\ V(\pi,y) &= 0, \quad 0 \le y \le a, \\ V(x,0) &= 0, \quad 0 \le x \le \pi. \end{aligned}$$

Use the method of separation of variables to show that $V(x,y)$ can be written in the form

$$V(x,y) = \sum_{m=1}^{\infty} C_m \sinh(my) \sin(mx) \ .$$

[15 marks]

Find the coefficients C_m given that the boundary condition at $y = a$ is

$$V(x,a) = x(\pi - x), \quad 0 \le x \le \pi \ .$$

[10 marks]

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6. (a) Find the eigenvalues and eigenfunctions of the boundary value problem

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0 \ , \quad X(0) = X(\pi) = 0 \ \ ,$$

where $\alpha \neq 0$.

[10 marks]

(b) The flow of heat in a thin bar of length π is governed by the heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \ ,$$

and satisfies the boundary conditions

$$u(0,t) = u(\pi,t) = 0 \ ,$$

together with the initial condition

$$u(x,0) = \frac{1}{3}\sin(3x) + \sin(5x) \ ,$$

and some time decay as t tends to infinity

$$\lim_{t \to +\infty} u(x,t) = 0 \ , \text{ for every } x \in [0,\pi] \ .$$

Show, using separation of variables, that

$$u(x,t) = \frac{1}{3}\sin(3x)e^{-9\kappa t} + \sin(5x)e^{-25\kappa t} \ .$$

[15 marks]