PAPER CODE NO.
MATH283

THE UNIVERSITY
of LIVERPOOL

ANSWERS TO SEPTEMBER 2002 EXAMINATION

Bachelor of Engineering: Year 2
Bachelor of Science: Year 2
Master of Engineering: Year 2
Master of Physics: Year 2

FIELD THEORY AND PARTIAL DIFFERENTIAL EQUATIONS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Attempt FOUR questions only. All questions are of equal value.

In this paper $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ represent unit vectors parallel to the x, y and z axes respectively.

Comments:

There are only FIVE questions in this exam paper which was set in september 2002 (resit). Note that you will have SIX questions in January 2005, but you should make an attempt to FOUR questions only. The first question of next january exam looks similar to question 1 of the present paper. Question 2(a) of next exam is analogous to question 2(a) of the present paper. The next two questions of january exam look like questions 2(b) and 3 of the present paper. Finally, the last two questions of january exam look like questions 4 and 5 of the present paper.

1. (a) [*This exercise is a really good training for exercise* **1** *of January 2005 exam. Do not forget to revise the notion of directional derivative as well.*]

Given that

$$\phi(x, y, z) = x^2 + 6y^2 + z^2 \quad ,$$

its gradient $\nabla\phi$ is :

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (2x, 12y, 2z) \ .$$

The unit *outward* normal to the ellipsoid $\phi(x, y, z) = x^2 + 6y^2 + z^2 = 6$ at a general point (x, y, z) is (note that the sign is positive)

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{\mid \nabla\phi \mid} = \frac{(2x, 12y, 2z)}{\sqrt{4x^2 + 144y^2 + 4z^2}} = \frac{(x, 6y, z)}{\sqrt{x^2 + 36y^2 + z^2}} \ .$$

At point $\mathbf{a} = (1, 1, 1)$ it can be simplified as:

$$\hat{\mathbf{n}} = \frac{(1, 6, 1)}{\sqrt{38}} \ .$$

Hence, the cartesian equation for the tangent plane which touches the ellipsoid at that point is given by

$$\begin{aligned} &(\mathbf{r} - \mathbf{a}) \cdot \hat{\mathbf{n}} = 0 \\ &(\mathbf{r} - (1, 1, 1)) \cdot \frac{(1, 6, 1)}{\sqrt{38}} = 0 \\ &(x - 1) + 6(y - 1) + (z - 1) = 0 \end{aligned}$$

Or

$$x + 6y + z = 8 \ .$$

(b) We first write

$$\begin{aligned} \nabla \cdot (r^2\mathbf{r}) &= (\partial/\partial x, \partial/\partial y, \partial/\partial z) \cdot \left(r^2(x, y, z)\right) \\ &= \frac{\partial}{\partial x}\left(xr^2\right) + \frac{\partial}{\partial y}\left(yr^2\right) + \frac{\partial}{\partial z}\left(zr^2\right) \\ &= x\frac{\partial}{\partial x}\left(r^2\right) + r^2\frac{\partial x}{\partial x} + y\frac{\partial}{\partial y}\left(r^2\right) + r^2\frac{\partial y}{\partial y} \\ &\quad + z\frac{\partial}{\partial z}\left(r^2\right) + r^2\frac{\partial z}{\partial z} \ . \end{aligned}$$

Now, we note that

$$\frac{\partial}{\partial x}(x^2 + y^2 + z^2) = 2x \ .$$

A similar result holds for the partial derivatives with respect to y and z.
Therefore, we obtain

$$\begin{aligned} \nabla \cdot (r^2 \mathbf{r}) &= 2x^2 + r^2 + 2y^2 + r^2 + 2z^2 + r^2 \\ &= 2r^2 + 3r^2 \\ &= 5r^2 \,. \end{aligned}$$

2. [*The part (a) of this exercise looks like the part (a) of exercise* **2** *of January 2005 exam. The part (b) of this exercise looks like the part (a) of exercise* **3** *of January 2005 exam. The best way to prepare the parts (b) of exercises* **2** *and* **3** *of January 2005 exam is to revise the exercises on conservative fields (vector fields whose curl is zero) and double integrals over circle and ellipsis (polar coordinates).*]

(a) We want to evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr} \ ,$$

where

$$\mathbf{F} = (2xy + xyz)\hat{\mathbf{i}} + (x^2 + xz)\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

and the curve $\mathcal{C}$ is the straight line starting at the point $(0,0,0)$ and finishing at $(1,1,1)$.

For this, we first parametrise the path $\mathcal{C}$ in 3D space by the equations

$$\begin{aligned} x &= t \\ y &= t \\ z &= t \ , \end{aligned}$$

where t starts at 0 and ends at 1.

We then need to express the vector field $\mathbf{F}$ in terms of the parameter t

$$\mathbf{F}(t) = (2t^2 + t^3)\hat{\mathbf{i}} + (t^2 + t^2)\hat{\mathbf{j}} + t^2\hat{\mathbf{k}} \ .$$

Further, if we consider the parametrised vector

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} = t\hat{\mathbf{i}} + t\hat{\mathbf{j}} + t\hat{\mathbf{k}} \ ,$$

its total derivative writes as

$$\frac{d\mathbf{r}}{dt} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \ .$$

Altogether, the line integral of the vector field $\mathbf{F}$ over the path $\mathcal{C}$ is expressed as

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr} &= \int_0^1 \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \\ &= \int_0^1 \left(2t^2 + t^3, 2t^2, t^2\right) \cdot (1,1,1) \ dt \\ &= \int_0^1 \left(5t^2 + t^3\right) \ dt \\ &= \left[\frac{5}{3}t^3 + \frac{1}{4}t^4\right]_0^1 \\ &= \frac{5}{3} + \frac{1}{3} = \frac{23}{12} \ . \end{aligned}$$

(b) Let us state Gauss's theorem for a differentiable vector field $\mathbf{F}$ defined over a volume τ with bounding surface S.

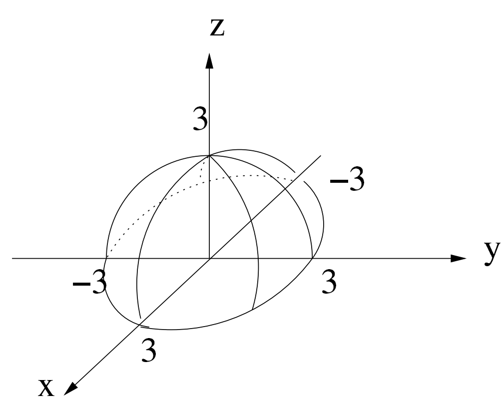
Gauss's (or divergence's) theorem :

Given a volume τ which is bounded by a piecewise continuous surface S, and a vector function $\mathbf{F}$ which is continuous and has continuous partial derivatives in τ, then

$$\iiint_\tau \nabla \cdot \mathbf{F} d\tau = \oint_S \mathbf{F} \cdot d\mathbf{S} \quad . \tag{1}$$

Notice that the surface integral S has been drawn with a circle around it, in order to indicate that this surface is *closed*, i.e. that it entirely encompasses the volume τ. Also, $d\mathbf{S}$ is defined as the product of a small area (let's say $dxdy$) by the unit outward normal $\hat{\mathbf{n}}$ to the surface S.

The region τ is the hemi-spherical volume which is enclosed by the surface $x^2 + y^2 + z^2 = 9$ and the plane $z = 0$, and lies above the (x, y) plane. Hence, it looks as follows (a hat!): Using Gauss's



theorem, the surface integral can be expressed as

$$\begin{aligned}
& \oint_S \left[2x\hat{\mathbf{i}} - 3x^2yz^2\hat{\mathbf{j}} + x^2z^3\hat{\mathbf{k}}\right] \cdot d\mathbf{S} \\
= & \iiint_\tau \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left[2x\hat{\mathbf{i}} - 3x^2yz^2\hat{\mathbf{j}} + x^2z^3\hat{\mathbf{k}}\right] d\tau \\
= & \iiint_\tau (2 - 3x^2z^2 + 3x^2z^2)\, d\tau \\
= & \; 2\,(\text{Volume of region } \tau) \\
= & \; 2\frac{4}{3}\frac{\pi 3^3}{2} = 36\pi \; .
\end{aligned}$$

3. [*Same spirit as part (a) of exercise* **4** *in January 2005 exam. The best way to prepare the part (b) of exercise* **4** *set in January, is to look at the exercise on Stokes's theorem in set 5 of lecture notes.*]

Let us first state the Stokes' theorem for a differentiable vector field $\mathbf{F}$ defined over a surface S bounded by a closed curve $\mathcal{C}$.
Stokes' theorem:
Given a surface S which is bounded by a piecewise continuous curve $\mathcal{C}$, and a vector function $\mathbf{F}$ which is continuous and has continuous partial derivatives on S, then

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \tag{2}$$

Now, the curl of the vector field

$$\mathbf{F} = 5y\hat{\mathbf{i}} + 4x\hat{\mathbf{j}} + 3z\hat{\mathbf{k}} \; .$$

writes as

$$\begin{aligned} \nabla \times \mathbf{F_1} &= (0-0)\,\hat{\mathbf{i}} - (0-0)\,\hat{\mathbf{j}} + \hat{\mathbf{k}}\,(4-5) \\ &= -\hat{\mathbf{k}} \; . \end{aligned}$$

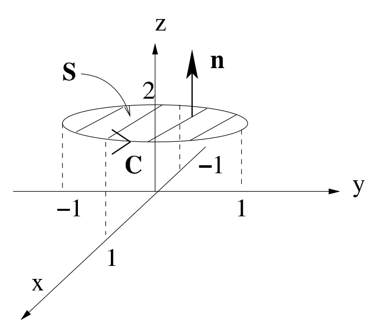
Let us now evaluate the surface integral

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where S_1 is the plane surface bounded by the circular path

$$x^2 + y^2 = 1 \quad , \qquad z = 2 \quad .$$

which is depicted on the figure below so that the unit normal $\hat{\mathbf{n}}$ to the surface $\mathbf{S}$ is simply the vector of the canonical basis $\hat{\mathbf{k}}$ (we deduce the orientation of $\hat{\mathbf{n}}$ from the thumb rule). Thus, an infinitesimal element



$d\mathbf{S}$ of orientable surface $\mathbf{S}$ writes as $d\mathbf{S} = \hat{\mathbf{n}}dxdy = \hat{\mathbf{k}}dxdy$ and we end up with

$$\begin{aligned} \int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int\int_S (-\hat{\mathbf{k}}) \cdot \hat{\mathbf{k}}dxdy = -\int\int_S dxdy \\ &= -\,(\text{area of } S) \\ &= -\pi \; . \end{aligned}$$

Let us now evaluate the path integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \ ,$$

where $\mathcal{C}$ is the boundary of the surface S above, traversed in the counterclockwise direction.

First of all we use the parametrisation

$$\begin{aligned} x &= \cos t \\ y &= \sin t \\ z &= 2 \end{aligned}$$

with

$$0 \leq t < 2\pi \quad .$$

Then

$$\mathbf{F}(t) = 5 \sin t \hat{\mathbf{i}} + 4 \cos t \hat{\mathbf{j}} + 6 \hat{\mathbf{k}} \quad ,$$

also

$$\mathbf{r}(t) = \cos t \hat{\mathbf{i}} + \sin t \hat{\mathbf{j}} + 2 \hat{\mathbf{k}}$$

hence

$$\frac{d\mathbf{r}}{dt} = -\sin t \hat{\mathbf{i}} + \cos t \hat{\mathbf{j}} + 0 \hat{\mathbf{k}} \quad .$$

Therefore, the integral is

$$\begin{aligned} \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left(-5 \sin^2 t + 4 \cos^2 t \right) dt \\ &= \int_0^{2\pi} \left(4 - 9 \sin^2 t \right) dt \\ &= \left(\int_0^{2\pi} 4 dt - 9 \int_0^{2\pi} \frac{1 - \cos(2t)}{2} dt \right) \\ &= [4t]_0^{2\pi} - \frac{9}{2} \left[t - \frac{\sin(2t)}{2} \right]_0^{2\pi} \\ &= 8\pi - 9\pi = -\pi \ . \end{aligned} \tag{3}$$

Hence, we have checked that

$$\int \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = -\pi = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \ . \tag{4}$$

4. [*The exercise provides a good training for exercise* **5** *of January 2005 exam. You should ALSO look at the exercise on Laplace's equation given in the set 7 of lecture notes. Do not forget that you should attempt ONLY FOUR exercises at the exam in january 2005. If you find it hard to work with hyperbolic sine and cosine functions, have a look at the set 6 of lecture notes (you may skip this question and focus on question* **5** *which only involve sine, cosine and exponential functions.)*]

Let us show by direct substitution that the general solution to the differential equation

$$\frac{d^2X}{dx^2} - \alpha^2 X = 0, \quad \alpha \neq 0,$$

is

$$X = A\cosh(\alpha x) + B\sinh(\alpha x) \quad .$$

First, we note that

$$\cosh(\alpha x) = \frac{e^{\alpha x} + e^{-\alpha x}}{2} \ , \ \sinh(\alpha x) = \frac{e^{\alpha x} - e^{-\alpha x}}{2} \ .$$

From this we see that

$$\frac{d}{dx}\cosh(\alpha x) = \alpha\sinh(\alpha x) \ , \ \frac{d}{dx}\sinh(\alpha x) = \alpha\cosh(\alpha x) \ .$$

Therefore, it follows that

$$\frac{dX}{dx} = A\alpha\sinh(\alpha x) + B\alpha\cosh(\alpha x) \ .$$

Altogether, we obtain

$$\begin{aligned}
\frac{d^2X}{dx^2} &= \frac{d^2}{dx^2}\Big(A\cosh(\alpha x) + B\sinh(\alpha x)\Big) \\
&= \alpha\frac{d}{dx}\Big(A\sinh(\alpha x) + B\cosh(\alpha x)\Big) \\
&= \alpha^2\Big(A\cosh(\alpha x) + B\sinh(\alpha x)\Big) \\
&= \alpha^2 X \ .
\end{aligned}$$

Now, we want to solve Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad (5)$$

within the region $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ and with the boundary conditions

$$V(x,0) = V(x,\pi) = 0 \ , \qquad (6)$$
$$V(0,y) = 0 \quad , \quad V(\pi,y) = -1/4\sin(3y) \ . \qquad (7)$$

Firstly we try a solution of the form

$$V(x,y) = X(x)Y(y) \quad .$$

There's no guarantee that a solution of this form will exist, let alone satisfy all the boundary conditions, however if we can find one which does then by uniqueness it must be the correct solution.

Substituting this trial form of the solution (the german word is *ansatz*) into Laplace's equation, we obtain

$$\frac{\partial^2}{\partial x^2}\left(X(x)Y(y)\right) + \frac{\partial^2}{\partial y^2}\left(X(x)Y(y)\right) = 0 \quad .$$

Or,

$$X''(x)Y(y) + Y''(y)X(x) = 0 \quad ,$$

where I've used the primes (" $'$ ") to indicate the *full* derivative with respect to the relevant variable.

We can re-arrange this equation into the form

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

and so we've *separated* the variables x and y into different parts of the differential equation. In other words, we've turned an equation with both variables x and y into an equation which is a function of x *plus* a function of y.

Now the only way you can have two functions of different variables adding together to give zero is if *they're both equal to a constant.*

That is,

$$\frac{X''(x)}{X(x)} = \pm\alpha^2 = -\frac{Y''(y)}{Y(y)} \quad . \tag{8}$$

Here I've written the constant as '$\pm\alpha^2$' rather than, say 'α' or 'κ' in order to emphasise that there is a definite choice of sign to be made here. The constant can, at this stage, be either poisitive or negative. If the constant is positive (we choose $+\alpha^2$) then we find that we have to solve the pair of ordinary differential equations

$$\frac{X''(x)}{X(x)} = +\alpha^2 \quad , \quad \frac{Y''(y)}{Y(y)} = -\alpha^2 \quad . \tag{9}$$

This choice will result in a combination of cosh and sinh functions for $X(x)$, whereas $Y(y)$ will be made up of a combination of cos and sin functions. If, however, we choose the constant to be negative (we choose $-\alpha^2$ in (8)), then the differential equations which we end up having to solve are

$$\frac{X''(x)}{X(x)} = -\alpha^2 \quad , \quad \frac{Y''(y)}{Y(y)} = +\alpha^2 \quad . \tag{10}$$

This choice results in a combination of cos and sin functions for $X(x)$, and $Y(y)$ is then made up of cosh and sinh functions.

It looks like the choice of sign of α will have a radical effect on the solution to the problem. Clearly one choice is 'right', and the other is 'wrong'. How do we tell which is which?

The answer is that one of the choices will be incapable of satisfying the boundary conditions. We saw in the last section that any combination of sinh's and cosh's will be unable to make up a function which shrinks to zero at two points. The boundary conditions for this problem imply that $Y(0) = Y(\pi) = 0$ and so we *cannot* end up with a solution of the form $Y(y) = A\cosh(\alpha y) + B\sinh(\alpha y)$ - this would imply that $Y(y) = 0$. (If you don't believe this, then you should go ahead and try it. Even if you make the 'wrong' choice for α^2 then you'll find that after a few lines you're unable to satisfy the first of the boundary conditions. The choice for the sign of α^2 isn't a drastic decision, on which the whole method hangs. If you've made a mistake then you'll know it pretty soon).

When in doubt, then, we can look for places where the function vanishes at two points. In this case we notice that the solution vanishes when $y = 0$ and $y = L$. We know then that the function $Y(y)$ must satisfy the differential equation

$$\frac{d^2Y}{dy^2} = -\alpha^2 Y \tag{11}$$

(if it didn't, then it would have to satisfy the other equation, which means that it would be zero) and the function $X(x)$ must satisfy the differential equation

$$\frac{d^2X}{dx^2} = \alpha^2 X \tag{12}$$

We know that the general solution to (11) is

$$Y(y) = A\cos(\alpha y) + B\sin(\alpha y) \quad , \tag{13}$$

and the boundary conditions, which come from (6), give us the boundary conditions for Y:

$$Y(0) = Y(\pi) = 0 \quad . \tag{14}$$

Substituting $Y(0) = 0$ into the general solution, we obtain

$$0 = A\cos 0 + B\sin 0 \quad ,$$

or

$$A = 0 \ \ .$$

Substituting the second boundary condition $y(\pi) = 0$, we obtain

$$0 = B \sin \alpha\pi$$

which implies that

$$\alpha\pi = n\pi \ , \quad \text{where } n = 0, 1, 2, \ldots$$

that is, $\alpha = n$ where n is some positive integer. We can now write down the *eigenfunctions* of the differential equation (11). These are

$$Y_n(y) = B_n \sin(nx) \tag{15}$$

and there are an infinity of these, one for each value of n.

We now turn to the solution of the second differential equation (12). The general solution to this equation is

$$X(x) = C \cosh(\alpha x) + D \sinh(\alpha x) \tag{16}$$

and we now attempt to evaluate as much of this as possible, given the information we have. The boundary conditions which we wish to apply are

$$V(0, y) = 0 \ \ , \quad V(\pi, y) = -\frac{1}{4} \sin(3y) \tag{17}$$

The boundary condition on the right involves y, and so isn't much help, since it doesn't allow us to say much about the function $X(x)$ which depends on the variable x only. The first boundary condition doesn't depend on y at all, so we can say that this implies

$$X(0) = 0 \ \ .$$

If we look back at the properties of sinh and cosh (detailed in the last chapter) then we notice that $\cosh(0) = 1$ always, whereas $\sinh(0) = 0$. This means that there can be no cosh functions at all in the solution; if they were there, then the function $X(x)$ wouldn't vanish when $x = 0$. The solution to the second differential equation is then

$$X(x) = D \sinh(\alpha x) \ \ . \tag{18}$$

Or, since we already have a restriction on α (it must be equal to some integer n) we can write

$$X(x) = D \sinh(nx) \ \ . \tag{19}$$

Note that we haven't used 'symmetry' properties or anything like that to eliminate the cosh functions. The requirement that $X(x)$ vanish when $x = 0$ was enough.

We now combine the two parts of our solution:

$$\begin{aligned} V_n(x,y) &= X(x)Y_n(y) \\ &= D\sinh(nx)B_n\sin(ny) \\ &= C_n\sinh(nx)\sin(ny) \end{aligned} \tag{20}$$

where we have combined the two constants D and B_n into a new constant we we call C_n. Again there are an infinite number of these solutions, one for each value of n. By the principle of superposition we can add any of these together, so the general solution is

$$V(x,y) = \sum_{n=0}^{\infty} C_n \sinh(nx)\sin(ny) \tag{21}$$

By picking the right constants C_n we can ensure that the last remaining boundary condition is satisfied. This is

$$-\frac{1}{4}\sin(3y) = V(\pi, y)$$

So we require that

$$-\frac{1}{4}\sin(3y) = \sum_{n=0}^{\infty} C_n \sinh(n\pi)\sin(ny) \tag{22}$$

In this case the choice is pretty simple: We know that all the $\sin(nx)$ functions are *orthogonal* (we'll look at a more strict definition of this in the next example), and so all the sin terms on the left and right hand sides have to match up. The only function appearing on the left-hand side of (22) is $\sin(3y)$, and so all the C_n terms on the right-hand side have to be zero except for C_3, which is given by

$$-\frac{1}{4} = C_3\sinh(3\pi) \tag{23}$$

In other words,

$$C_n = \begin{cases} -\frac{1}{4}\frac{1}{\sinh(3\pi)} & \text{for} \quad n = 3 \\ 0 & \text{for} \quad n \neq 3 \end{cases} \tag{24}$$

The full solution to the differential equation is then

$$\begin{aligned} V(x,y) &= \sum_{n=0}^{\infty} C_n \sinh(nx)\sin(ny) \\ &= -\frac{1}{4}\frac{1}{\sinh(3\pi)}\sinh(3x)\sin(3y) \quad . \end{aligned} \tag{25}$$

5. [*The exercise provides a good training for exercise* **6** *of January 2005 exam. You should ALSO look at the exercise on the Heat equation given in the set 7 of lecture notes. Do not forget that you should attempt ONLY FOUR exercises at the exam in january 2005. This exercise only involves sine, cosine and exponential functions, so it may be regarded as easier than the exercise* **4** *for some of you.)*]

(a) To solve the equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \tag{26}$$

we try a solution of the form

$$u(x,t) = X(x)T(t) \tag{27}$$

Substituting (27) into (26) leads to

$$X(x)T'(t) = \kappa X''(x)T(t) \ , \tag{28}$$

which can be recast as

$$\frac{X''(x)}{X(x)} = \frac{1}{\kappa}\frac{T'(t)}{T(t)} \ , \tag{29}$$

[*Here, we have used the fact that* $X(x)$ *and* $T(t)$ *are* $\neq 0$, *since we are looking for non-trivial solutions of the spectral problems in* X *and* T.]

Hence, we must solve

$$\frac{X''(x)}{X(x)} = -\alpha^2 \ , \ \frac{1}{\kappa}\frac{T'(t)}{T(t)} = -\alpha^2 \ . \tag{30}$$

[*Anticipating the exponential solution in* $T(t)$, *we have picked a negative separation constant* $-\alpha^2$, *so that the solution remains finite when* t *tends to infinity.*]

To find the eigenvalues and eigenfunctions of the boundary value problem

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0 \ , \quad X(0) = X(\pi) = 0 \ \ , \tag{31}$$

we proceed as follows:

We first notice that the general solution to:

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0 \ ,$$

is

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x) \ .$$

When $x = 0$, this simplifies to:

$$X(0) = A\cos(\alpha 0) + B\sin(\alpha 0) = A$$

which thanks to boundary condition $X(0) = 0$ implies

$$A = 0 \ .$$

Now, looking at the other extremity of the bar ($X(\pi) = 0$) gives:

$$B \sin(\alpha\pi) = 0 \ .$$

So that we are left with:

$$\alpha := \alpha_n = n \ , \ n = 0, \ 1, \ 2, ...$$

Thus, the eigenvalues of the spectral problem (31) look like:

$$\alpha_n = n \ , \ n = 0, \ 1, \ 2, ... \tag{32}$$

and the associated eigenfunctions can be written as:

$$X_n(x) = B_n \sin(\alpha_n x) \ .$$

The eigenfunctions associated with the first ordinary differential equation in (30) subject to boundary conditions $X(0) = X(\pi) = 0$ are

$$X_n(x) = B_n \sin(nx)$$

The second equation in (30) is

$$T'(t) = -\alpha^2 \kappa T(t) \ ,$$

which has the general solution

$$T_n(t) = C_n \exp(-\alpha_n^2 \kappa t) \ .$$

(b) We can write the general solution $u(x,t)$ as

$$\begin{aligned} u(x,t) &= \sum_0^\infty X_n(x) T_n(t) \\ &= \sum_0^\infty B_n \sin(nx) \, C_n \exp(-\alpha_n^2 \kappa t) \\ &= \sum_0^\infty \Big[B_n \sin(nx) \\ &\qquad \times C_n \exp\left(-n^2 \kappa t\right) \Big] \ , \end{aligned} \tag{33}$$

where in the last equation we have used (32).

We know that at $t = 0$

$$u(x,0) = \sum_{m=1}^{\infty} \frac{(-1)^m \sin(mx)}{m} \ .$$

We thus deduce from (33) that

$$\sum_{m=1}^{\infty} \frac{(-1)^m \sin(mx)}{m} = \sum_{0}^{\infty} B_n \, C_n \sin(nx) \ . \tag{34}$$

We note that for $n = 0$, $\sin(nx)$ is zero, thus the series on the right side starts at $n = 1$. In this case the choice of coefficients is pretty simple: We know that all the $\sin(nx)$ functions are *orthogonal* (we'll look at a more strict definition of this in the next example), and so all the sin terms on the left and right hand sides have to match up. The functions appearing on both sides of (34) do match for any $\sin(nx)$, and so

$$B_n C_n = \frac{(-1)^n}{n} \ . \tag{35}$$

The full solution to the differential equation is then

$$\begin{aligned} V(x,y) &= \sum_{n=1}^{\infty} B_n C_n \sin(nx) e^{-n^2 \kappa t} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n \sin(nx) e^{-n^2 \kappa t}}{n} \ . \end{aligned} \tag{36}$$