PAPER CODE NO.
MATH283

THE UNIVERSITY
of LIVERPOOL

SEPTEMBER 2002 EXAMINATIONS

Bachelor of Engineering: Year 2
Bachelor of Science: Year 2
Master of Engineering: Year 2
Master of Physics: Year 2

FIELD THEORY AND PARTIAL DIFFERENTIAL EQUATIONS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Attempt FOUR questions only. All questions are of equal value.

In this paper $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ represent unit vectors parallel to the x, y and z axes respectively.

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1. (a) Given that

$$\phi(x,y,z) = x^2 + 6y^2 + z^2 \quad ,$$

find $\nabla\phi$. Hence calculate the outward pointing unit normal to the ellipsoid

$$x^2 + 6y^2 + z^2 = 8$$

at the point $(1,1,1)$. Use this to find the cartesian equation of the tangent plane at this point.

[17 marks]

(b) Calculate the divergence of the vector function

$$\mathbf{v} = r^2\,\mathbf{r} \quad ,$$

where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $r = |\mathbf{r}|$. [8 marks]

2. (a) Evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{dr} \quad ,$$

where

$$\mathbf{F} = (2xy + xyz)\hat{\mathbf{i}} + (x^2 + xz)\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

and the curve $\mathcal{C}$ is the straight line starting at the point $(0,0,0)$ and finishing at $(1,1,1)$. [10 marks]

(b) State Gauss's theorem for a differentiable vector field $\mathbf{F}$ defined over a volume τ with bounding surface S.

[5 marks]

The region τ is the hemi-spherical volume which is enclosed by the
surface $x^2 + y^2 + z^2 = 9$ and the plane $z = 0$, and lies above the (x,y) plane.

Sketch τ and, using Gauss's theorem or otherwise, evaluate the surface integral

$$\int\!\!\int_S (2x\hat{\mathbf{i}} - 3x^2yz^2\hat{\mathbf{j}} + x^2z^3\hat{\mathbf{k}})\cdot d\mathbf{S}$$

where S is the outwards-oriented bounding surface of τ. [10 marks]

3. State Stokes' theorem for a differentiable vector field $\mathbf{F}$ over a surface S bounded by a closed curve $\mathcal{C}$.

[5 marks]

Calculate the curl of the vector field

$$\mathbf{F} = 5y\hat{\mathbf{i}} + 4x\hat{\mathbf{j}} + 3z\hat{\mathbf{k}} \ .$$

[5 marks]

Evaluate the surface integral

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where S is the upwards-oriented plane surface bounded by the circular path

$$x^2 + y^2 = 1, \qquad z = 2.$$

[7 marks]

Evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \ ,$$

where C is the boundary of the surface S above, traversed in the counter-clockwise direction. (Hint: you may use the parametrisation $x = \cos t$, $y = \sin t$, $z = 2$, where $0 \leq t < 2\pi$). Hence verify Stokes' theorem in this case. [8 marks]