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Model solution

Question 1.

(a) $\underline{E} = -\nabla\phi$

$$\phi(x, y, z) = \sqrt{y^2+z^2} - x.$$

(i) $$\underline{E} = -\nabla\phi = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$= -\left(-1,\ \tfrac{1}{2}(y^2+z^2)^{-1/2}\cdot 2y,\ \tfrac{1}{2}(y^2+z^2)^{-1/2}\cdot 2z\right)$$

$$= \left(1, \frac{-y}{\sqrt{y^2+z^2}}, \frac{-z}{\sqrt{y^2+z^2}}\right)$$

[5 marks]

(ii) $\underline{v} = \underline{i} + 3\underline{k} = (1, 0, 3)$

Point $= (1, 2, -4)$

$$D_{\underline{v}} = \nabla\phi \cdot \frac{\underline{v}}{|\underline{v}|}$$

$$= \left(-1, \frac{+y}{\sqrt{y^2+z^2}}, \frac{+z}{\sqrt{y^2+z^2}}\right) \cdot \frac{(1,0,3)}{\sqrt{10}}$$

$$= \frac{1}{\sqrt{10}}\left(-1 + \frac{3z}{\sqrt{y^2+z^2}}\right)$$

Hence at $(1, 2, -4)$

$$D_{\underline{v}} = \frac{1}{\sqrt{10}}\left(-1 - \frac{12}{\sqrt{20}}\right)$$

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Model solution (Cont'd).

Question 1 (Cont'd).

(ii) (Cont'd)

$$= \frac{1}{\sqrt{10}}\left(\frac{-\sqrt{20} - 12}{\sqrt{20}}\right)$$

$$= \frac{-2\sqrt{5} - 12}{\sqrt{200}}$$

$$= \frac{-2\sqrt{5} - 12}{10\sqrt{2}} = \frac{-\sqrt{5} - 6}{5\sqrt{2}}$$

[5 marks]

(iii). Max. D $= |\nabla\phi|_{(1,2,-4)}$

$$= \sqrt{1 + \frac{4}{20} + \frac{16}{20}}$$

$$= \sqrt{2}$$

Direction is $\nabla\phi_{(1,2,-4)}$

$$= \left(-1, \frac{2}{\sqrt{20}}, -\frac{4}{\sqrt{20}}\right)$$

$$= \left(-1, \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$$

[5 marks]

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Model solution (Cont'd).

Question 1. (Cont'd)

(b).

$$\underline{v} = (x^2y - 2z)\underline{i} + (2x^3z^2 + y)\underline{j} + (3xy + 2z^2)\underline{k}.$$

(i)

$$\nabla \cdot \underline{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(x^2y - 2z,\ 2x^3z^2 + y,\ 3xy + 2z^2\right)$$

$$= 2xy + 1 + 4z$$

$$= \underline{\underline{2xy + 4z + 1.}}$$

[5 marks]

(ii)

$$\nabla \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y - 2z & 2x^3z^2 + y & 3xy + 2z^2 \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial}{\partial y}(3xy + 2z^2) - \frac{\partial}{\partial z}(2x^3z^2 + y)\right) - \underline{j}\left(\frac{\partial}{\partial x}(3xy + 2z^2) - \frac{\partial}{\partial z}(x^2y - 2z)\right)$$

$$+ \underline{k}\left(\frac{\partial}{\partial x}(2x^3z^2 + y) - \frac{\partial}{\partial y}(x^2y - 2z)\right)$$

$$= \underline{\underline{\underline{i}(3x - 4x^3z) - \underline{j}(3y + 2) + \underline{k}(6x^2z^2 - x^2)}}$$

[5 marks]

[All covered in lectures and class work]

(4).

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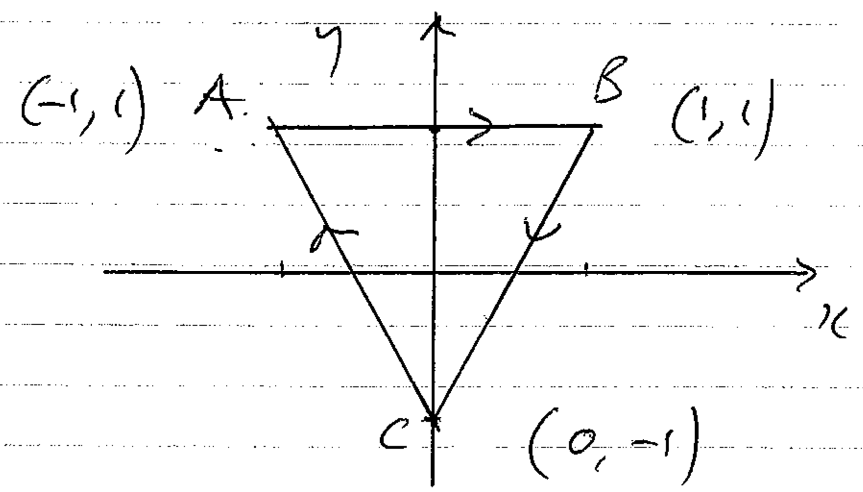
Model solutions (Cont'd)

Question 2

(a).

$$\underline{F} = (y-x)\,\underline{i} + xy\,\underline{j}.$$

C = triangle with vertices $(0,-1)$, $(1,1)$, $(-1,1)$ clockwise so is



Split the integral into three parts

$$\int_C \underline{F}\cdot d\underline{r} = \int_{AB} \underline{F}\cdot d\underline{r} + \int_{BC} \underline{F}\cdot d\underline{r} + \int_{CA} \underline{F}\cdot d\underline{r}$$

For AB we have $y = 1$ and x goes from -1 to 1. Hence choose parameter $t \in [-1, 1]$ with

$$x = t$$
$$y = 1.$$

$$\frac{d\underline{r}}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j}$$
$$= \underline{i} = (1, 0)$$

so

$$\underline{F}\cdot\frac{d\underline{r}}{dt} = (1-t,\ t)\cdot(1, 0) = 1-t$$

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Model solutions (Cont'd)

Question 2. (Cont'd)

(a) (Cont'd)

Hence $\int_{AB} \underline{F} \cdot d\underline{r} = \int_{AB} \underline{F} \cdot \frac{d\underline{r}}{dt}\, dt$

$$= \int_{t=-1}^{t=1} (1-t)\, dt$$

$$= \left[t - \frac{t^2}{2} \right]_{-1}^{1}$$

$$= \left(1 - \frac{1}{2}\right) - \left(-1 - \frac{1}{2}\right)$$

$$= 2.$$

For BC, x goes from 1 to 0 and y from 1 to -1. This is the line with slope 2 and intercept $y = -1$ so its Cartesian equation is $y = 2x - 1$.

Hence choose parameter t going from 1 to 0 with

$$x = t$$

$$y = 2t - 1$$

$$\frac{d\underline{r}}{dt} = 1\underline{i} + 2\underline{j} = (1, 2)$$

so

$$\underline{F} \cdot \frac{d\underline{r}}{dt} = (t-1,\ 2t^2 - t) \cdot (1, 2)$$

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Model solution (Cont'd)

Question 2 (Cont'd).

(a) (Cont'd)

$$= t - 1 + 4t^2 - 2t$$
$$= 4t^2 - t - 1$$

Hence
$$\int_{BC} \underline{F} \cdot d\underline{r} = \int_{BC} \underline{F} \cdot \frac{d\underline{r}}{dt}\, dt$$

$$= \int_{t=1}^{t=0} (4t^2 - t - 1)\, dt$$

$$= \left[\frac{4t^3}{3} - \frac{t^2}{2} - t \right]_1^0$$

$$= (0) - (4/3 - 1/2 - 1)$$

$$= - \frac{8 - 3 - 6}{6}$$

$$= \frac{1}{6}$$

For CA x goes from 0 to -1 and y from -1 to 1. This is the line with slope -2 and intercept -1 so its Cartesian equation is $y = -2x - 1$.

Hence choose parameter t going from 0 to -1

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Model solutions (Cont'd).

Question 2 (Cont'd).

(a) (Cont'd).

with

$$x = t$$
$$y = -2t - 1$$

$$\frac{d\underline{r}}{dt} = 1\underline{i} - 2\underline{j} = (1, -2)$$

so

$$\underline{F} \cdot \frac{d\underline{r}}{dt} = (-3t - 1, -2t^2 - t) \cdot (1, -2)$$
$$= -3t - 1 + 4t^2 + 2t$$
$$= 4t^2 - t - 1$$

Hence

$$\int_{CA} \underline{F} \cdot d\underline{r} = \int_{CA} \underline{F} \cdot \frac{d\underline{r}}{dt}\, dt$$

$$= \int_{t=0}^{t=-1} (4t^2 - t - 1)\, dt$$

$$= \left[\frac{4t^3}{3} - \frac{t^2}{2} - t \right]_0^{-1}$$

$$= \left(-\tfrac{4}{3} - \tfrac{1}{2} + 1\right) - (0)$$

$$= \frac{-8 - 3 + 6}{6} = -\frac{5}{6}$$

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Model Solutions (Cont'd)

Question 2 (Cont'd)

(a) (Cont'd)

Hence

$$\int_C \underline{F} \cdot d\underline{r} = 2 + \frac{1}{6} - \frac{5}{6}$$

$$= 2 - \frac{2}{3}$$

$$= \frac{4}{3}$$

[15 marks]

(b). $\underline{F} = (6x^2 + 3z)\,\underline{i} + \underline{j} + 3x\,\underline{k}$

For $\underline{F}$ to be a gradient we need curl $\underline{F}$ $= \nabla \times \underline{F} = \underline{0}$

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6x^2+3z & 1 & 3x \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial}{\partial y}(3x) - \frac{\partial}{\partial z}(1)\right) - \underline{j}\left(\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial z}(6x^2+3z)\right) + \underline{k}\left(\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(6x^2+3z)\right)$$

$$= \underline{i}(0-0) - \underline{j}(3-3) + \underline{k}(0-0) = \underline{0}$$

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Model solutions (Cont'd).

Question 2 (Cont'd).

(b) (Cont'd)

Hence $\underline{F}$ can be expressed as the gradient of a scalar field since $\nabla \times (\nabla \phi) = \underline{0}$ for any scalar field ϕ.

For the field, we need

$$\frac{\partial \phi}{\partial x} = 6x^2 + 3z \qquad (1)$$

$$\frac{\partial \phi}{\partial y} = 1 \qquad (2)$$

$$\frac{\partial \phi}{\partial z} = 3x \qquad (3).$$

simultaneously.

Integrating these equations we have, from (1)

$$\phi_x = 2x^3 + 3xz + f_x(y, z)$$

from (2)

$$\phi_y = y + f_y(x, z)$$

from (3)

$$\phi_z = 3xz + f_z(x, y)$$

Hence, by inspection, the required scalar field is

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(10)

M283. Jan 07 exam
Model solution (Cont'd)

Question 2 (Cont'd)
(b) (Cont'd).

$$\phi = 2x^3 + 3xz + y + C$$

where C is an arbitrary constant.

Check

$$\nabla\phi = \left(6x^2 + 3z,\ 1,\ 3x\right) = \underline{F} \checkmark$$

(10 marks)

[All covered in lectures and classwork]

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Model solutions (Cont'd).

Question 3.

(a) Gauss's Theorem relates a volume integral over an enclosed region to a surface integral over the surface enclosing the region.

Formally, it states:-

Given a volume, τ, which is bounded by a piecewise continuous surface, S, and a vector function, $\underline{F}$, which is continuous and has continuous partial derivatives in τ, then

$$\int_\tau \nabla \cdot \underline{F} \, d\tau = \oint_S \underline{F} \cdot d\underline{S}$$

Here the left hand integral is the volume integral of the divergence of $\underline{F}$ and the right hand integral is the surface integral denoting the flux of $\underline{F}$ through the surface. Note that the surface integral has a circle on the integral sign to denote that the surface is closed (ie it completely encompasses the volume) [10 marks]

(b). Using Gauss's Theorem, the surface integral

$$\iint_S (2\underline{i} + 2y\underline{j} + z\underline{k}) \cdot \hat{\underline{n}} \, dS$$

can be written as

$$\oint_S (2\underline{i} + 2y\underline{j} + z\underline{k}) \cdot d\underline{S} = \int_\tau \nabla \cdot (2\underline{i} + 2y\underline{j} + z\underline{k}) \, d\tau$$

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Model solutions (Cont'd).

Question 3 (Cont'd).

(b) (Cont'd)

$$= \int_\tau \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(2, 2y, z)\, d\tau$$

$$= \int_\tau (0 + 2 + 1)\, d\tau$$

$$= \int_\tau 3\, d\tau.$$

$$= 3\int_\tau d\tau$$

Now $\int_\tau d\tau$ is just the volume of the region enclosed by the ellipsoid

$$3x^2 + y^2 + 4z^2 = 8.$$

Writing this in standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

we have

$$\frac{x^2}{8/3} + \frac{y^2}{8} + \frac{z^2}{2} = 1$$

so $a^2 = 8/3$, $b^2 = 8$ and $c^2 = 2$

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Model solutions (Cont'd).

Question 3 (Cont'd)

(b) (Cont'd)

and hence

$$a = \frac{2\sqrt{2}}{\sqrt{3}}, \quad b = 2\sqrt{2}, \quad c = \sqrt{2}$$

Hence the volume of the region is

$$\tfrac{4}{3}\pi abc = \tfrac{4}{3}\pi \times \frac{2\sqrt{2}}{\sqrt{3}} \times 2\sqrt{2} \times \sqrt{2}$$

$$= \frac{32\sqrt{2}}{3\sqrt{3}}\pi$$

Hence $3\int_\tau d\tau = \dfrac{32\sqrt{2}}{\sqrt{3}}\pi.$

Hence

$$\iint_S \left(2\underline{i} + 2y\underline{j} + z\underline{k}\right)\cdot\hat{\underline{n}}\, dS = \frac{32\sqrt{2}}{\sqrt{3}}\pi.$$

[15 marks]

[All covered in lectures and classwork)

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Model solutions (Cont'd)

Question 4

(a). Stokes' Theorem relates a surface integral over a bounded surface to a line integral along the boundary of the surface.

Formally, it states:-

Given a surface, S, which is bounded by a continuous curve, C, and a vector function, $\underline{F}$, which is continuous and has continuous partial derivatives on S, then

$$\int_S (\nabla \times \underline{F}) \cdot d\underline{S} = \oint_C \underline{F} \cdot d\underline{r}$$

[5 marks]

(b). $\underline{F} = x^2y\,\underline{i} + 3xy\,\underline{j} + 4y\,\underline{k}$

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 3xy & 4y \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial}{\partial y}(4y) - \frac{\partial}{\partial z}(3xy)\right) - \underline{j}\left(\frac{\partial}{\partial x}(4y) - \frac{\partial}{\partial z}(x^2y)\right) + \underline{k}\left(\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(x^2y)\right)$$

$$= \underline{i}(4-0) - \underline{j}(0-0) + \underline{k}(3y - x^2)$$

$$= (4, 0, 3y - x^2)$$

[5 marks]

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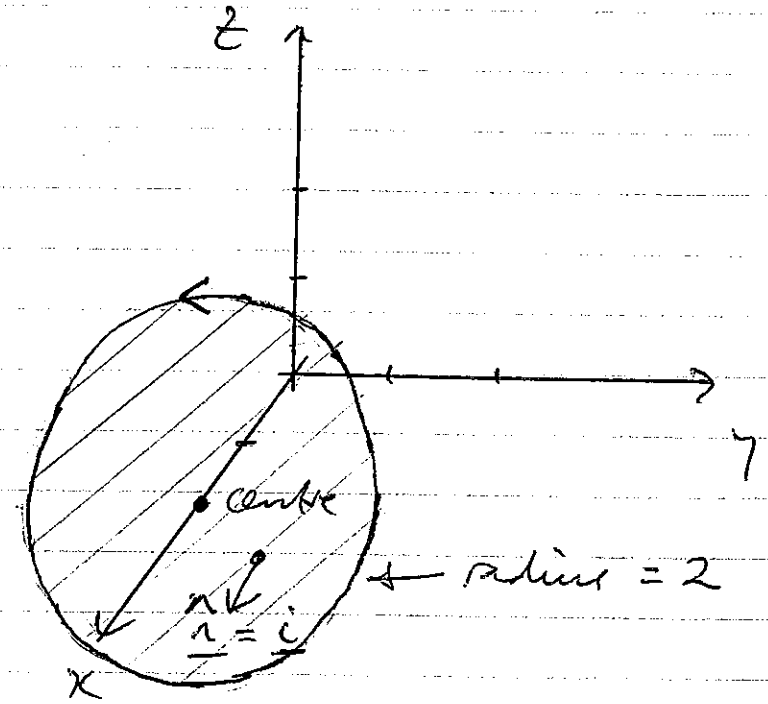
Model solutions (Cont'd).

Question 4 (Cont'd).

(c). The surface is thus



$$d\underline{S} = \hat{\underline{n}}\, dy\, dz = \underline{i}\, dy\, dz$$

Hence

$$\int_S (\nabla \times \underline{F}) \cdot d\underline{S}$$

$$= \int_S (4, 0, 3y - x^2) \cdot (1, 0, 0)\, dy\, dz$$

$$= \int_S 4\, dy\, dz = 4 \int_S dy\, dz$$

Now $\int_S dy\, dz$ is just the area of the surface

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Model solution (Cont'd).

Question 4 (Cont'd)

(c) (Cont'd)

so

$$4\int_S dy\,dz = 4 \times \pi \times 2^2 = 16\pi$$

Hence

$$\iint_S (\nabla \times \underline{F}) \cdot d\underline{S} = 16\pi.$$

(7 marks)

(d). $\underline{F} = (x^2y,\ 3xy,\ 4y)$

C = circle $y^2+z^2=4$ in $x=2$ plane.

Parametrize the path with parameters

$x=2,\ y = 2\cos t,\ z = 2\sin t$

Hence, in terms of t, we have

$$\underline{F} = (8\cos t,\ 12\cos t,\ 8\cos t)$$

and

$$\frac{d\underline{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (0,\ -2\sin t,\ 2\cos t)$$

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Model solutions (Cont'd)

Question 4 (Cont'd)

(d) (Cont'd)

Hence

$$\int_C \underline{F} \cdot d\underline{r} = \int_C \underline{F} \cdot \frac{d\underline{r}}{dt}\, dt$$

$$= \int_{t=0}^{t=2\pi} (8\cos t, 12\cos t, 8\cos t) \cdot (0, -2\sin t, 2\cos t)\, dt$$

$$= \int_0^{2\pi} (-24 \sin t \cos t + 16\cos^2 t)\, dt$$

$$= \int_0^{2\pi} \left(-12\sin(2t) + 16\left(\frac{1 + \cos(2t)}{2}\right)\right) dt$$

$$= \Big[6\cos(2t) + 8t + 4\sin(2t)\Big]_0^{2\pi}$$

$$= (6\cos(4\pi) + 16\pi + 4\sin(4\pi)) - (6\cos 0 + 0 + 4\sin 0)$$

$$= 6 + 16\pi - 6$$

$$= 16\pi.$$

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Model solution (Cont'd)
Question 4 (Cont'd).

Hence $\iint_S (\nabla \times \underline{F}) \cdot d\underline{S} = \int_C \underline{F} \cdot d\underline{r}$

and so Stokes's Theorem is verified in this case [8 marks]

[All covered in lectures and class work]

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Model solution (Cont'd)

Question 5

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$0 \le x \le a, \quad 0 \le y \le \pi.$$

$$V(0, y) = 0, \quad 0 \le y \le \pi$$

$$V(x, 0) = 0, \quad 0 \le x \le a$$

$$V(x, \pi) = 0, \quad 0 \le x \le a.$$

(a). Let $V(x,y) = X(x)\,Y(y)$.

Then we have

$$\frac{\partial V}{\partial x} = X'Y \qquad \frac{\partial V}{\partial y} = XY'$$

$$\frac{\partial^2 V}{\partial x^2} = X''Y \qquad \frac{\partial^2 V}{\partial y^2} = XY''$$

Hence, in the PDE,

$$X''(x)\,Y(y) + X(x)\,Y''(y) = 0$$

$$\Rightarrow \quad X''(x)\,Y(y) = -X(x)\,Y''(y)$$

$$\Rightarrow \quad \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

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Model solutions (Contd)

Question 5 (Contd)

(a) (Contd)

Since the variables are separate, the two sides of this equation must be equal to some constant, say $\pm\alpha^2$, with the sign to be chosen.

Hence we have two ODE's

$$\frac{d^2X}{dx^2} = \pm\alpha^2 X \qquad (\alpha \neq 0)$$

and

$$\frac{d^2Y}{dy^2} = \mp\alpha^2 Y \qquad (\alpha \neq 0)$$

To choose the sign for α^2, we look at the boundary conditions.

$$V(x, 0) = 0, \qquad 0 \leq x \leq a$$

and

$$V(x, \pi) = 0, \qquad 0 \leq x \leq a$$

imply that

$$Y(0) = Y(\pi) = 0$$

Hence $Y(y)$ cannot be made up of cosh and sinh functions since $\sinh(\pi) \neq 0$, $\cosh(\pi) \neq 0$, $\sinh(0) = 0$ and $\cosh(0) = 1 \neq 0$.

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Model solutions (Cont'd).

Question 5 (Cont'd)

(a) (Cont'd).

Hence the equation for $Y(y)$ must be of the form

$$\frac{d^2Y}{dy^2} = -\alpha^2 Y \quad \text{———} \quad (1)$$

so that its solution is made up of sin and cos terms.

Hence the equation for $X(x)$ is of the form

$$\frac{d^2X}{dx^2} = \alpha^2 X \quad \text{———} \quad (2).$$

The general solution of (1) is

$$Y(y) = A\cos(\alpha y) + B\sin(\alpha y).$$

Now using the boundary condition $Y(0) = 0$ we have

$$Y(0) = 0 = A\cos 0 + B\sin 0$$

$$\Rightarrow \quad A = 0$$

From the boundary condition $Y(\pi) = 0$ we have

$$Y(\pi) = 0 = B\sin(\alpha\pi)$$

so, for $B \neq 0$, we need $\alpha\pi = 0, \pi, 2\pi \ldots$ so $\alpha_n = n$, where $n = 1, 2 \ldots$

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Model solutions (Cont'd)

Question 5 (Cont'd)

(a) (Cont'd)

Hence the eigen functions of the ODE for Y are

$$Y_n(y) = B_n \sin(ny) \qquad n = 1, 2, \ldots$$

The general solution of (2) is

$$X(x) = C \cosh(\alpha x) + D \sinh(\alpha x)$$

From the boundary condition $V(0, y) = 0$ we have $X(0) = 0$ so

$$X(0) = 0 = C \cosh(0) + D \sinh(0)$$

$$\Rightarrow C = 0$$

Hence the general solution is

$$X(x) = D \sinh(\alpha x)$$

$$\Rightarrow X_n(x) = D_n \sinh(nx), \quad n = 1, 2, \ldots$$

Hence, for $V_n(x, y)$, we have

$$V_n(x, y) = X_n(x)\, Y_n(y)$$

$$= D_n \sinh(nx)\, B_n \sin(ny)$$

$$= E_n \sinh(nx) \sin(ny)$$

Hence, using the principle of superposition, the general solution of the PDE is

M283. Jan 07 exam.

Model solutions (Cont'd).

Question 5 (Cont'd)

(a) (Cont'd)

$$V(x, y) = \sum_{n=1}^{\infty} E_n \sinh(nx) \sin(ny)$$

[15 marks]

(b). $V(a, y) = y$

$$\Rightarrow \quad y = \sum_{n=1}^{\infty} E_n \sinh(na) \sin(ny) \quad \text{--- (3)}$$

Using the orthogonality of $\sin(ny)$ we multiply both sides of (3) by $\sin(ky)$ and integrate from 0 to π to give

$$\int_0^{\pi} y \sin(ky)\, dy =$$

$$\sum_{n=1}^{\infty} E_n \sinh(na) \int_0^{\pi} \sin(ny) \sin(ky)\, dy$$

$$= E_k \sinh(ka)\, \pi/2$$

For the integral on the left hand side we need to use parts

with $u = y$ and $\frac{dv}{dy} = \sin(ky)$

M283. Jan 07 exam. Model solution (Cont'd). 1/11/06

Question 5 (Contd) (b) (Contd)

we have

$$v = -\frac{\cos(ky)}{k}, \quad \frac{du}{dy} = 1$$

so

$$\int y \sin(ky)\,dy = -y\frac{\cos(ky)}{k} + \int \frac{\cos(ky)}{k}\,dy$$

$$= -y\frac{\cos(ky)}{k} + \frac{\sin(ky)}{k^2}$$

so

$$\int_0^\pi y\sin(ky)\,dy = \left[-y\frac{\cos(ky)}{k} + \frac{\sin(ky)}{k^2}\right)_0^\pi$$

$$= \left(\left(-\pi\frac{\cos(k\pi)}{k} + \frac{\sin(k\pi)}{k^2}\right) - (0+0)\right)$$

$$= -\frac{\pi}{k}\cos(k\pi)$$

Hence we have

$$-\frac{\pi}{k}\cos(k\pi) = \frac{\pi}{2}E_k\sinh(ka)$$

$$\Rightarrow \quad E_k = -\frac{2}{k}\frac{\cos(k\pi)}{\sinh(ka)}$$

If k is even, $\cos(k\pi) = 1$ and if k is odd $\cos(k\pi) = -1$, so

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Model solutions (Cont'd)

Question 5 (Cont'd)

(b) (Cont'd)

$$E_k = \frac{-2(-1)^k}{k \sinh(ka)} = \frac{2(-1)^{k+1}}{k \sinh(ka)}$$

Hence

$$E_n = \frac{2(-1)^{n+1}}{n \sinh(na)}$$

Hence, substituting in the solution from part (a), we have

$$V(x, y) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sinh(nx) \sin(ny)}{n \sinh(na)}$$

[10 marks]

[All covered in lectures and class work].

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M283. Jan 07 exam.
Model solution (Cont'd).

Question 6.

(a). $$\frac{d^2X}{dx^2} + \alpha^2 X = 0.$$

$$X(0) = X(L) = 0, \qquad \alpha \neq 0.$$

Equation is standard and solution is

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x)$$

Using the boundary condition $X(0) = 0$ we have

$$X(0) = 0 = A\cos 0 + B\sin 0$$

$$\Rightarrow \quad A = 0.$$

Hence
$$X(x) = B\sin(\alpha x).$$

Now using the boundary condition $X(L) = 0$ we have

$$X(L) = 0 = B\sin(\alpha L)$$

$$\Rightarrow \quad \alpha L = n\pi, \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow \quad \alpha = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

Hence the eigenvalues are
$$\alpha_n = \frac{n\pi}{L}, \qquad n = 1, 2, 3, \ldots$$

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Model solution (Cont'd)

Question 6 (Cont'd).

(a) (Cont'd)

and the associated eigen functions are

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$$

[10 marks]

(b).
$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = \sin(2x) + \frac{1}{5}\sin(3x).$$

$$\lim_{t\to\infty} u(x,t) = 0 \quad \forall\, x \in [0, L]$$

Let $u(x,t) = X(x)\,T(t)$.

$$\frac{\partial u}{\partial x} = X'T$$

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

$$\frac{\partial u}{\partial t} = XT'$$

Hence, substituting in the PDE

$$XT' = KX''T$$

$$\Rightarrow \quad \frac{X''(x)}{X(x)} = \frac{1}{K}\frac{T'(t)}{T(t)}$$

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Question 6 (Cont'd)

(b) (Cont'd)

Since the variables are separate, the two sides of this equation must be equal to some constant, say $\pm\alpha^2$, with the sign to be chosen.

Hence

$$\frac{X''}{X} = \pm\alpha^2, \qquad \frac{1}{\kappa}\frac{T'}{T} = \pm\alpha^2$$

Since the solution decays with time we anticipate an exponential solution for $T(t)$ so choose $-\alpha^2$ giving the ODE's

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0 \qquad (1)$$

and

$$\frac{1}{\kappa}\frac{dT}{dt} + \alpha^2 T = 0 \qquad (2)$$

Equation (2) is standard and its general solution is

$$T_n(t) = C_n e^{(-\alpha_n^2 \kappa t)}$$

Hence, using the result of part (a), the general solution of the problem is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) C_n e^{(-\alpha_n^2 \kappa t)}$$

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Model solution (Cont'd)

Question 6 (Cont'd).

(b) (Cont'd).

$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) C_n e^{\left(-\frac{n^2\pi^2}{L^2} K t\right)}$$

Now $u(x,0) = \sin(2x) + \frac{1}{5}\sin(3x)$ so

$$u(x,0) = \sin(2x) + \frac{1}{5}\sin(3x) = \sum_{n=1}^{\infty} B_n C_n \sin\left(\frac{n\pi x}{L}\right)$$

From orthogonality of sine functions, $B_n C_n = 0$ for every n different from 2 and 3. Also, $B_2 C_2 = 1$ and $B_3 C_3 = 1/5$.

Hence we have

$$u(x,t) = \sin\left(\frac{2\pi x}{L}\right) e^{\left(-\frac{4\pi^2}{L^2} K t\right)} + \frac{1}{5}\sin\left(\frac{3\pi x}{L}\right) e^{\left(\frac{-9\pi^2}{L^2} K t\right)}$$

[15 marks]

[All covered in lectures and class work]