PAPER CODE NO.
MATH 283

THE UNIVERSITY
of LIVERPOOL

JANUARY EXAMINATIONS 2007

Bachelor of Engineering : Year 2
Bachelor of Science : Year 1
Bachelor of Science : Year 2
Master of Engineering : Year 2
Master of Mathematics : Year 2
Master of Physics : Year 2

FIELD THEORY AND PARTIAL DIFFERENTIAL EQUATIONS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Attempt FOUR questions only. All questions are of equal value (25 marks each). Throughout the paper $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ represent unit vectors parallel to the x, y and z axes respectively.

1. (a) An electric field $\mathbf{E}$ has a scalar potential function ϕ where $\mathbf{E} = -\nabla\phi$. You are given that

$$\phi(x, y, z) = \sqrt{y^2 + z^2} - x.$$

(i) Calculate the electric field $\mathbf{E}$.

(ii) Find the directional derivative of ϕ at the point $(1, 2, -4)$ in the direction of the vector $\hat{\mathbf{i}} + 3\hat{\mathbf{k}}$.

(iii) What is the maximum directional derivative of ϕ at the point $(1, 2, -4)$? In which direction does this occur? [15 marks]

(b) For the vector field

$$\mathbf{u} = (x^2y - 2z)\hat{\mathbf{i}} + (2x^3z^2 + y)\hat{\mathbf{j}} + (3xy + 2z^2)\hat{\mathbf{k}},$$

calculate

(i) the divergence, $\nabla \cdot \mathbf{u}$, of $\mathbf{u}$,

(ii) the curl, $\nabla \times \mathbf{u}$, of $\mathbf{u}$. [10 marks]

2. (a) Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where

$$\mathbf{F} = (y - x)\hat{\mathbf{i}} + xy\hat{\mathbf{j}}$$

and C is the triangle with vertices $(0, -1), (1, 1)$ and $(-1, 1)$ traversed in a clockwise direction. [15 marks]

(b) Show that the vector field

$$\mathbf{F} = (6x^2 + 3z)\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3x\hat{\mathbf{k}}$$

can be expressed as the gradient of a scalar field and find that field.

[10 marks]

3. (a) State Gauss's Theorem for a differentiable vector field $\mathbf{F}$ defined over a volume τ with bounding surface S, clearly explaining the meaning of the terms in the theorem. [10 marks]

(b) The region τ is the ellipsoid enclosed by the surface $3x^2 + y^2 + 4z^2 = 8$. Use Gauss's Theorem to evaluate

$$\iint_S (2\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\ dS,$$

where S is the bounding surface and $\hat{\mathbf{n}}$ the outward unit normal to S. [15 marks]

4. (a) State Stokes's Theorem for a differentiable vector field $\mathbf{F}$ defined over a surface S bounded by a closed curve C. [5 marks]

(b) Calculate the curl of the vector field

$$\mathbf{F} = x^2y\hat{\mathbf{i}} + 3xy\hat{\mathbf{j}} + 4y\hat{\mathbf{k}}.$$

[5 marks]

(c) Evaluate, by direct integration, the surface integral

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} d\mathbf{S},$$

where S is the disc

$$y^2 + z^2 \leq 4,$$

in the $x = 2$ plane, oriented by $\hat{\mathbf{n}} = \hat{\mathbf{i}}$. [7 marks]

(d) Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the boundary of the surface S, traversed in the clockwise direction looking from the origin (i.e. the conventional direction using the right hand rule with $\hat{\mathbf{n}} = \hat{\mathbf{i}}$). (Hint: you may use the parameters $y = 2\cos t, z = 2\sin t, x = 2$, where $0 \leq t \leq 2\pi$).

Hence verify Stokes's theorem in this case. [8 marks]

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5. A scalar function $V(x,y)$ obeys Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad (1)$$

in the region $(0 \le x \le a), (0 \le y \le \pi)$, together with the boundary conditions

$$V(0,y) = 0,\ 0 \le y \le \pi, \qquad (2)$$
$$V(x,0) = 0,\ 0 \le x \le a, \qquad (3)$$
$$V(x,\pi) = 0,\ 0 \le x \le a. \qquad (4)$$

(a) Use the method of separation of variables to show that $V(x,y)$ can be written in the form

$$V(x,y) = \sum_{n=1}^{\infty} E_n \sinh(nx)\sin(ny).$$

[15 marks]

(b) Find the coefficients E_n given that the boundary condition at $x = a$ is

$$V(a,y) = y,\ 0 \le y \le \pi,$$

and show that $V(x,y)$ can be written as

$$V(x,y) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sinh(nx)\sin(ny)}{n\sinh(na)}.$$

[Hint: you may assume that $\int_0^{\pi} \sin(ny)\sin(ky)dy = \pi/2$, if $n = k$, $n \ne 0$, and 0 otherwise.]

[10 marks]

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6. (a) Find the eigenvalues and eigenfunctions of the boundary value problem

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0, \qquad X(0) = X(L) = 0,$$

where $\alpha \neq 0$. [10 marks]

(b) The flow of heat in a thin bar of length L is governed by the heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

and satisfies the boundary conditions

$$u(0, t) = u(L, t) = 0$$

together with the initial condition

$$u(x, 0) = \sin(2x) + \frac{1}{5}\sin(3x),$$

and some time decay as t tends to infinity, so that

$$\lim_{t\to+\infty} u(x, t) = 0, \quad \text{for every } x \in [0, L].$$

Show, using separation of variables and the result of part (a), that

$$u(x,t) = \sin\left(\frac{2\pi x}{L}\right) e^{\left(-4\frac{\pi^2}{L^2}\kappa t\right)} + \frac{1}{5}\sin\left(\frac{3\pi x}{L}\right) e^{\left(-9\frac{\pi^2}{L^2}\kappa t\right)}.$$

[Hint: you may assume the orthogonality of the basis of sine functions so that, from any expression of the form $a\sin(bx) + c\sin(dx) = \sum_{n=1}^{\infty} A_n \sin(nx)$, it follows that $A_n = 0$ for every n different from b and d and $A_b = a$ and $A_d = c$.]
[15 marks]

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Correction

Question 4 (c)

The "S" in "dS" in the integral should not be in bold face type. The integral should be

$$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, dS$$