# Answers for Exam of January 2006

[M283 - Field theory and PDEs]

**1.** (a) [*Standard exercise, similar to lectures and tutorials.*]
Gradient:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (2x, 4y, 2z)\ .$$

**2 marks**
Directional derivative of $\Phi$:

$$\mathcal{D}_{\mathbf{b}}\Phi = \nabla\Phi \cdot \frac{\mathbf{b}}{\mid \mathbf{b} \mid} = (2x, 4y, 2z) \cdot \frac{(1,0,0)}{\sqrt{1}}\ .$$

**2 marks**
At point $\mathbf{a} = (2,1,1)$:

$$\mathcal{D}_{\mathbf{b}}\Phi(\mathbf{a}) = \nabla\Phi(\mathbf{a}) \cdot \frac{\mathbf{b}}{\mid \mathbf{b} \mid} = 4\ .$$

**1 marks**
Normal to ellipsoid:

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{\mid \nabla\phi \mid} = \frac{(x, 2y, z)}{\sqrt{x^2 + 4y^2 + z^2}}\ .$$

**3 marks**
At point $\mathbf{a} = (2,1,1)$:

$$\hat{\mathbf{n}} = \frac{(2,2,1)}{\sqrt{9}}\ .$$

**1 mark**
Cartesian equation of tangent plane:

$$(\mathbf{r} - \mathbf{a}) \cdot \hat{\mathbf{n}} = 0$$
$$2(x-2) + 2(y-1) + (z-1) = 0$$

**3 marks**
Or

$$2x + 2y + z = 7\ .$$

**1 mark**

(b) [*Bookwork i.e similar to lectures, tutorials and homeworks*].
From definition of curl

$$\nabla \times (\frac{\mathbf{v}}{\phi}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(\phi^{-1}v_x, \phi^{-1}v_y, \phi^{-1}v_z\right) \tag{1}$$

**1 mark**
So that,

$$\begin{aligned}\nabla \times \left(\frac{\mathbf{v}}{\phi}\right) &= \phi^{-2}\left\{\phi\left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{i}} - \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)\hat{\mathbf{j}}\right.\right. \\ &+ \left.\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{k}}\right] + \left[\left(v_y\frac{\partial \phi}{\partial z} - v_z\frac{\partial \phi}{\partial y}\right)\hat{\mathbf{i}}\right. \\ &- \left.\left.\left(v_x\frac{\partial \phi}{\partial z} - v_z\frac{\partial \phi}{\partial x}\right)\hat{\mathbf{j}} + \left(v_x\frac{\partial \phi}{\partial y} - v_y\frac{\partial \phi}{\partial x}\right)\hat{\mathbf{k}}\right]\right\} \\ &= \frac{\phi\nabla \times \mathbf{v} + \mathbf{v} \times \nabla\phi}{\phi^2} = \frac{\phi\nabla \times \mathbf{v} - \nabla\phi \times \mathbf{v}}{\phi^2} \qquad (2)\end{aligned}$$

**3 marks**
[*This is a standard exercise on gradient and curl discussed in lectures (see chapter 3 of lecture notes) and tutorials 2 and 3.*] We check that

$$\begin{aligned}\nabla\left(\frac{1}{r}\right) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2\right)^{-1/2} \\ &= -\frac{1}{r^3}\mathbf{r}\ ,\ r \neq 0\ ,\end{aligned}$$

**2 marks**
and

$$\begin{aligned}\nabla(r^3) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2\right)^{3/2} \\ &= 3r\mathbf{r}\ ,\end{aligned}$$

**1 mark**
as well as

$$\begin{aligned}\nabla \times (\mathbf{r}) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (x, y, z) \\ &= \mathbf{0}\ ,\end{aligned}$$

**1 mark**
where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $r = |\mathbf{r}|$.

[*Standard exercise*] Therefore, we end up with

$$\begin{aligned}\nabla \times \nabla\left(\frac{1}{r}\right) &= \nabla \times \left(-\frac{\mathbf{r}}{r^3}\right) \\ &= -\frac{r^3(\mathbf{0}) - 3r\mathbf{r} \times \mathbf{r}}{r^6} = \mathbf{0}\ ,\ r \neq 0\ .\end{aligned}$$

**3 marks**
In fact, this result holds for any smooth scalar field $\phi$ as seen during the lectures [*see also exercise 2*]. **1 mark**

**2.** (a) [*Standard exercise, similar to lectures and tutorials.*]

Line integral of $\mathbf{F}$ over $\mathcal{C}_1$:

$$\begin{aligned}\int_{\mathcal{C}_1} \mathbf{F}\cdot\mathbf{dr} &= \int_0^{2\pi} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt \\ &= \int_0^{2\pi} \left(2\cos t + t, \cos t + 2t\sin^2 t, 2t^2\sin t\right)\cdot(-sint, 1, \cos t)\ dt \\ &= \int_0^{2\pi} t^2\sin(2t)\,dt + \int_0^{2\pi} t\,dt - \int_0^{2\pi} t\sin(t)\,dt \\ &= 2\pi\ .\end{aligned}$$

**7 marks**

(b) Using

$$\begin{aligned}\nabla\times\nabla\phi &= \left(\frac{\partial^2}{\partial y\partial z}\phi - \frac{\partial^2}{\partial z\partial y}\phi\right)\hat{\mathbf{i}} \\ &- \left(\frac{\partial^2}{\partial x\partial z}\phi - \frac{\partial^2}{\partial z\partial x}\phi\right)\hat{\mathbf{j}} \\ &+ \left(\frac{\partial^2}{\partial x\partial y}\phi - \frac{\partial^2}{\partial y\partial x}\phi\right)\hat{\mathbf{k}}\ ,\end{aligned}$$

**3 marks**

and

$$\frac{\partial^2}{\partial y\partial z} = \frac{\partial^2}{\partial z\partial y}\ ,$$

we conclude

$$\nabla\times\nabla\phi = \mathbf{0}\ .$$

**1 mark**

[*Standard exercise, cf. tutorial 2.*]

Curl of $\mathbf{F}$

$$\nabla\times\mathbf{F} = (4yz - 4yz)\hat{\mathbf{i}} - (0-0)\hat{\mathbf{j}} + (1-1)\hat{\mathbf{k}} = \mathbf{0}\ . \tag{3}$$

**5 marks**

We deduce that $\mathbf{F}$ can be expressed as the gradient of a scalar field $\phi$. Hence,

$$\mathbf{F} = (2x+y)\hat{\mathbf{i}} + (x + 2yz^2)\hat{\mathbf{j}} + 2y^2z\hat{\mathbf{k}} = \nabla\phi\ .$$

**1 mark**

So that

$$\begin{aligned}\phi &= x^2 + xy + f_1(y,z) \\ \phi &= xy + y^2z^2 + f_2(x,z) \\ \phi &= y^2z^2 + f_3(x,y)\ ,\end{aligned}$$

**2 marks**

where $f_1$, $f_2$ and $f_3$ are three arbitrary functions which we need specify.

By inspection, we end up with

$$\phi = x^2 + xy + y^2 z^2 + C \ ,$$

**2 marks**

where $C$ is an arbitrary constant.

(c) Line integral of $\mathbf{F}$ over $\mathcal{C}_2$:

$$\begin{aligned}
\int_{\mathcal{C}_2} \mathbf{F} \cdot \mathbf{dr} &= \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \\
&= \int_0^{2\pi} (2+t, 1, 0) \cdot (0, 1, 0) \ dt \\
&= \int_0^{2\pi} dt \\
&= 2\pi \ .
\end{aligned}$$

**3 marks**

[*Bookwork: Gradient theorem and conservative fields.*]

Since $\mathbf{F}$ derives from a scalar field $\phi$, the line integrals in (a) and (c) can be reformulated as

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_{\mathcal{C}} \nabla\phi \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \frac{d\phi}{dt} dt \\
&= \phi(1, 2\pi, 0) - \phi(1, 0, 0) \\
&= 2\pi \ .
\end{aligned}$$

**1 mark**

**3.** [*Bookwork, similar exercises were set in lectures, tutorials and homeworks.*

Let us state Gauss's theorem for a differentiable vector field $\mathbf{F}$ defined over a volume $\tau$ with bounding surface $S$.

**Gauss's (or divergence's) theorem :**

Given a volume $\tau$ which is bounded by a piecewise continuous surface $S$, and a vector function $\mathbf{F}$ which is continuous and has continuous partial derivatives in $\tau$, then

$$\int_\tau \nabla \cdot \mathbf{F} d\tau = \oint_S \mathbf{F} \cdot d\mathbf{S} \quad . \tag{4}$$

Notice that the surface integral $S$ has been drawn with a circle around it, in order to indicate that this surface is *closed*, i.e. that it entirely encompasses the volume $\tau$. Also, $d\mathbf{S}$ is defined as the product of a small area (let's say $dxdy$) by the unit outward normal $\hat{\mathbf{n}}$ to the surface $S$.

**5 marks**

The exercise now splits in two sections (which can be independently addressed).

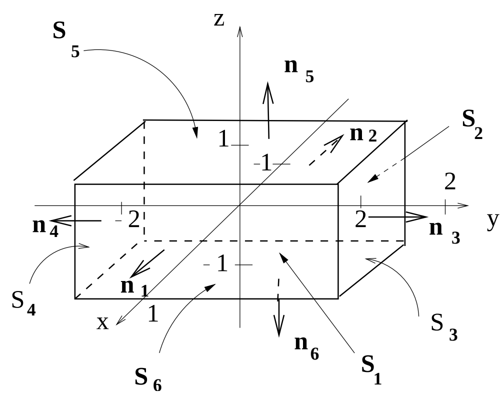


Figure 1: A rectangular box over which we apply Gauss's theorem.

(a) Divergence of $\mathbf{F}$:

$$\nabla \cdot \mathbf{F} = 1 + 4y + 3 \; .$$

**3 marks**

Volume integral of divergence of $\mathbf{F}$:

$$\int_\tau (1 + 4y + 3) d\tau = \int_{-1}^{1} dx \int_{-2}^{2} dy \int_{-1}^{1} dz (4 + 4y) = 32 \int_{-1}^{1} dx = 64 \quad . \tag{5}$$

**3 marks**

By Gauss's theorem, we deduce that

$$\oint_S \mathbf{F} \cdot d\mathbf{S} = \int_\tau (\nabla \cdot \mathbf{F}) d\tau = 64 \; ,$$

where $\tau$ is the interior of $S$.

Let us now check that this is indeed what we would get from the direct evaluation of the surface integral of $\mathbf{F}$ over the surface $\mathbf{S}$. For this, we decompose $\mathbf{S}$ into six surfaces $\mathbf{S}_1$, ..., $\mathbf{S}_6$ oriented by unit normals $\mathbf{n}_1$, ..., $\mathbf{n}_6$ as defined in Figure 1.

Let us first evaluate:

$$\begin{aligned}\int_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int_{S_1} (x, 2y^2, 3z) \cdot (1,0,0)\, dS \\ &= \int_{A_{yz}} x \frac{dydz}{(1,0,0)\cdot \hat{\mathbf{i}}} \\ &= x \int_{-2}^{2} dy \int_{-1}^{1} dz \\ &= 8\ ,\end{aligned}$$

since $x = 1$ on the surface $S_1$. **1 mark**

Similarly, we find

$$\int_{S_2} \mathbf{F} \cdot d\mathbf{S} = -8\ , \quad \text{and} \int_{S_3} \mathbf{F} \cdot d\mathbf{S} = 32\quad . \tag{6}$$

since $x = -1$ on the surface $S_2$ and $y = 2$ on $S_3$. **2 marks**

Also, we have

$$\int_{S_4} \mathbf{F} \cdot d\mathbf{S} = 32\ , \int_{S_5} \mathbf{F} \cdot d\mathbf{S} = 24\ , \quad \text{and} \int_{S_6} \mathbf{F} \cdot d\mathbf{S} = -24$$

since $y = -2$ on the surface $S_4$, $z = 1$ on $S_5$ and $z = -1$ on $S_6$. **3 marks**

Altogether, we obtain

$$\begin{aligned}\oint_S \mathbf{F} \cdot d\mathbf{S} &= \sum_{i=1}^{6} \int_{S_i} \mathbf{F} \cdot d\mathbf{S} \\ &= 64\ .\end{aligned}$$

We have therefore checked Gauss's theorem on this example. **1 mark**

(b) Applying Gauss's theorem, we derive that

$$\begin{aligned}\oint_S \mathbf{r} \cdot d\mathbf{S} &= \int_\tau \nabla \cdot \mathbf{r}\, d\tau \\ &= 3 \int_\tau d\tau = 3Vol(\tau)\ .\end{aligned} \tag{7}$$

**7 marks**

**4.** [*bookwork (same spirit as exercise* **3***).*]

Let us first state the Stokes' theorem for a differentiable vector field $\mathbf{F}$ defined over a surface $S$ bounded by a closed curve $\mathcal{C}$.

**Stokes' theorem**:

Given a surface $S$ which is bounded by a piecewise continuous curve $\mathcal{C}$, and a vector function $\mathbf{F}$ which is continuous and has continuous partial derivatives on $S$, then

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \tag{8}$$

**5 marks**

The exercise now splits into two sections (which can be independently addressed).

(a) Curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \left(z^2 + x\right)\hat{\mathbf{i}} - (0-0)\hat{\mathbf{j}} + \hat{\mathbf{k}}\,(-z-3) \ . \tag{9}$$

**3 marks**

Unit normal $\hat{\mathbf{n}}$ to the surface $\mathbf{S}$ (see Figure 2):

$$\hat{\mathbf{n}} = \frac{\nabla(x^2+y^2-2z)}{|\ \nabla(x^2+y^2-2z)\ |} = \frac{(x,y,-1)}{\sqrt{x^2+y^2+1}} \ .$$

(we deduce the orientation of $\hat{\mathbf{n}}$ from the thumb rule). **1 mark**

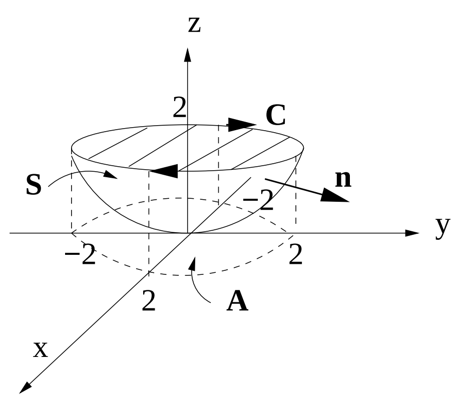


Figure 2: The surface of the paraboloid over which we apply Stokes' theorem.

Thus, an infinitesimal element $d\mathbf{S}$ of orientable surface $\mathbf{S}$ writes as $d\mathbf{S} = \hat{\mathbf{n}}\dfrac{dxdy}{|\ \hat{\mathbf{n}}\cdot\hat{\mathbf{k}}\ |}$ and we end up with

$$\begin{aligned}\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int\int_A (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\frac{dxdy}{|\ \hat{\mathbf{n}}\cdot\hat{\mathbf{k}}\ |} \\ &= \int\int_A \left[x\left(\frac{x^2+y^2}{2}\right)^2 + x^2 + \frac{x^2+y^2}{2} + 3\right] dxdy \ ,\end{aligned}$$

since $z = (x^2 + y^2)/2$ on the surface of the paraboloid.
Note that here, $A$ is the orthogonal projection of $\mathbf{S}$ onto the $(xy)$-plane as depicted on Figure 2. **3 marks**
Using polar coordinates, we obtain

$$\begin{aligned}\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \Big[(r\cos\theta)(r^4/2) \\ &+ r^2\cos^2\theta + r^2/2 + 3\Big]\, r dr d\theta = 20\pi \ .\end{aligned}$$

**2 marks**
Now, the integral of $\mathbf{F}$ over $\mathcal{C}$ is

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^{0} \left(-12\sin^2 t - 8\cos^2 t\right) dt = 20\pi \ . \tag{10}$$

**2 marks**
Hence, we have checked that

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 20\pi = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \ . \tag{11}$$

(b) Applying Stokes' theorem, we obtain

$$\oint_{\mathcal{C}} \nabla\phi \cdot d\mathbf{r} = \int\int_S (\nabla \times \nabla\phi) \cdot d\mathbf{S} = 0 \ , \tag{12}$$

since $\nabla \times \nabla\phi = \mathbf{0}$ for any (smooth) scalar function $\phi$ (see also ex. 2(b)).
**7 marks**

**5.** [*Standard exercise on Laplace equation i.e. classwork and homework.*]

(a) We try a solution which has the form

$$V(x,y) = X(x)Y(y) \quad , \tag{13}$$

which we can rearrange to get

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0 \quad . \tag{14}$$

**2 marks** which leads to (since $Y(0) = Y(L) = 0$)

$$\frac{d^2X}{dx^2} = +\alpha^2 X(x) \quad , \tag{15}$$

$$\frac{d^2Y}{dy^2} = -\alpha^2 Y(y) \quad . \tag{16}$$

**4 marks**

The general solution to the differential equation (16) is

$$Y(y) = C\cos(\alpha y) + D\sin(\alpha y) \tag{17}$$

**2 marks**

Using $Y(0) = Y(L) = 0$ we are left with

$$\alpha := \alpha_n = n\pi/L \qquad \text{where } n = 0, 1, 2, \ldots \tag{18}$$

**1 mark**

We now have the eigenvectors of the differential equation (16):

$$Y_n(y) = D_n \sin(n\pi y/L) \quad \text{, where } n = 0, 1, 2, \ldots \tag{19}$$

**1 mark**

The general solution to the first differential equation (15) is

$$X(x) = A\cosh(\alpha x) + B\sinh(\alpha x) \tag{20}$$

**2 marks**

Using $X(0) = 0$ , $X(L) = V_0$ we find that the eigenvectors of (15) are

$$X_n(x) = B_n \sinh(\alpha_n x) \ . \tag{21}$$

**2 marks**

(b) Combining the functions of $x$ and $y$, we obtain

$$V_n(x,y) = E_n \sinh(n\pi x/L)\sin(n\pi y/L) \ , \tag{22}$$

since we already know $\alpha_n = n\pi/L$ from (18).

**1 mark**

We now use the principle of superposition to write the general solution:

$$V(x,y) = \sum_{n=0}^{\infty} E_n \sinh(n\pi x/L)\sin(n\pi y/L) \quad . \tag{23}$$

**1 mark**

Now if we pick the coefficients $E_n$ correctly then we can satisfy the final boundary condition, which is

$$V(L,y) = V_0 \ .$$

So we want to collect coefficients $E_n$ such that

$$V_0 = \sum_{n=1}^{\infty} E_n \sinh(n\pi)\sin(ny) \ . \tag{24}$$

**1 mark**

We now multiply equation (24) by the function $\sin(k\pi y/L)$, and integrate it between 0 and $L$:

$$V_0 \int_0^L \sin(k\pi y/L)dy = E_k \sin(k\pi)\frac{L}{2} \ , \tag{25}$$

**1 mark**

where we have used the orthogonality relation (see Hint) to eliminate all terms but one from the infinite sum.

This leads to

$$E_k = \frac{2V_0}{\pi k}\frac{1-\cos(k\pi)}{\sinh(k\pi)} \ . \tag{26}$$

**1 mark**

We notice that $E_{2k} = 0$ and

$$E_{2k-1} = \frac{4V_0}{\pi}\frac{1}{(2k-1)\sinh((2k-1)\pi)} \ .$$

**1 mark**

Hence we conclude that

$$\begin{aligned} V(x,y) &= \sum_{n=1}^{\infty} E_n \sinh(n\pi x/L)\sin(n\pi y/L) \\ &= \frac{4V_0}{\pi}\sum_{n=1}^{\infty}\frac{\sinh((2n-1)\frac{\pi y}{L})}{(2n-1)\sinh((2n-1)\pi)}\sin((2n-1)\frac{\pi x}{L}) \ . \end{aligned} \tag{27}$$

**2 marks**

(c) *[Aim: To derive the famous Maximum Principle]* At the center of the square, the above expression reduces to

$$\begin{aligned} V(L/2, L/2) &= \frac{4V_0}{\pi} \sum_{n=1}^{\infty} \frac{\sinh((2n-1)\frac{\pi}{2})}{(2n-1)\sinh((2n-1)\pi)} \sin((2n-1)\frac{\pi}{2}) \\ &= \frac{2V_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)\cosh((2n-1)\pi/2)} \,, \end{aligned} \tag{28}$$

where we have used that $\sinh(2s) = 2\sinh s \cosh s$.

**3 marks**

Since $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + ...$ and $\cosh s \geq 1$, it follows that

$$0 \leq V(L/2, L/2) \leq \frac{V_0}{2} \leq V_0 \,.$$

**1 mark**

*Nota Bene*: This result holds true everywhere within the square thanks to the so-called maximum principle: The solution of Laplace equation takes its minimum and maximum values on the boundary of the domain.

**6.** [*Standard exercise on the wave equation, classwork and homework.*]

(a) The sepraration of variables leads to

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} \ . \tag{29}$$

**2 marks**

which by virtue of the boundary conditions $X(0) = X(a) = 0$ gives

$$\frac{d^2X}{dx^2} = -\alpha^2 X(x) \ , \tag{30}$$

$$\frac{d^2T}{dt^2} = -\alpha^2 T(t) \ . \tag{31}$$

**4 marks**

The general solution of (30) is:

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x) \ . \tag{32}$$

**1 mark**

Using $X(0) = 0$ and $X(a) = 0$ gives us a restraint on $\alpha$:

$$0 = B\sin(\alpha a) \ .$$

**1 mark**

That is,

$$\alpha = \frac{n\pi}{a} \quad \text{where } n = 0, 1, 2, \ldots \tag{33}$$

**1 mark**

We now have the eigenvalues of the differential equation (30):

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{a}\right) \quad \text{, where } n = 0, 1, 2, \ldots \tag{34}$$

**1 mark**

The general solution to the second differential equation (31) is

$$T(t) = C\cos(\alpha ct) + D\sin(\alpha ct) \ . \tag{35}$$

**1 mark**

From (33), we obtain:

$$T_n(t) = C_n \cos(\frac{n\pi ct}{a}) + D_n \sin(\frac{n\pi ct}{a}) \ . \tag{36}$$

**1 mark**

Combining the functions of $x$ and $t$ and using the principle of superposition we deduce the general solution:

$$V(x,t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\left[C_n \cos(\frac{n\pi ct}{a}) + D_n \sin(\frac{n\pi ct}{a})\right] . \tag{37}$$

**1 mark**

(b) To fix the constants, we should now use the initial boundary condition $V(x,0) = 0$ (the displacement of the string is initially zero) which implies that $B_n C_n = 0$. Hence, we deduce that

$$V(x,t) = \sum_{n=1}^{\infty} B_n D_n \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi ct}{a}\right) . \tag{38}$$

**2 marks**

Further, the velocity of the string at $t = 0$ is

$$U = \frac{\partial V}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n C_n \sin\left(\frac{n\pi x}{a}\right)\frac{n\pi c}{a} . \tag{39}$$

**1 mark**

Multiplying equation (39) by the function $\sin\frac{k\pi x}{a}$, and integrating it between 0 and $a$ we obtain:

$$U\int_0^a \sin\frac{k\pi x}{a}dx = \sum_{n=1}^{\infty} B_n C_n \left(\int_0^a \sin\frac{n\pi x}{a}\sin\frac{k\pi x}{a}\,dx\right)\frac{n\pi c}{a} = B_k C_k \frac{k\pi c}{2} \tag{40}$$

where we have used the orthogonality relation (see Hint) to eliminate all terms but one from the infinite sum.

**2 marks**

This leads to

$$B_k C_k = -2aU\frac{\cos(k\pi) - 1}{(k\pi)^2 c} . \tag{41}$$

**1 mark**

We now notice that $B_{2k}C_{2k} = 0$ and

$$B_{2k-1}C_{2k-1} = \frac{4aU}{\pi^2 c}\frac{1}{(2k-1)^2} .$$

We can now write down the full solution to the differential equation:

$$V(x,t) = \frac{4aU}{\pi^2 c}\sum_{n=1}^{\infty}\frac{\sin\frac{(2n-1)\pi x}{a}\sin\frac{(2n-1)\pi ct}{a}}{(2n-1)^2} . \tag{42}$$

**1 mark**

Using the trigonometric identity

$$\sin(pt)\sin(rt) = 1/2\left(\cos((p-r)t) - \cos((p+r)t)\right.\ ,$$

we obtain

$$\begin{aligned} V(x,t) = \ & \frac{2aU}{\pi^2 c}\left[\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left((2n-1)(x-ct)\frac{\pi}{a}\right)\right. \\ & \left. -\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left((2n-1)(x+ct)\frac{\pi}{a}\right)\right]. \end{aligned} \tag{43}$$

The physical interpretation is that the left series represents wave propagation to the right with speed $c$ and the right series represents wave propagation to the left with speed $c$.

**1 mark**