PAPER CODE NO.
**MATH283**

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# JANUARY 2006 EXAMINATIONS

Bachelor of Engineering: Year 2
Bachelor of Science: Year 2
Master of Engineering: Year 2
Master of Physics: Year 2

**FIELD THEORY AND PARTIAL DIFFERENTIAL EQUATIONS**

TIME ALLOWED : Two Hours

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INSTRUCTIONS TO CANDIDATES

Attempt FOUR questions only. All questions are of equal value (25 marks each).

Throughout the paper $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ represent unit vectors parallel to the $x$, $y$ and $z$ axes respectively.

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**1.** (a) Given that

$$\phi(x, y, z) = x^2 + 2y^2 + z^2 \quad ,$$

calculate $\nabla\phi$.

[2 marks]

Derive the expression for the directional derivative $\mathcal{D}_b\phi(\mathbf{a})$ of $\phi$ at point $\mathbf{a} = (2, 1, 1)$ in the direction of the vector $\mathbf{b} = (1, 0, 0)$.

[3 marks]

Calculate the outward unit normal to the ellipsoid

$$x^2 + 2y^2 + z^2 = 7$$

at the point $\mathbf{a} = (2, 1, 1)$.

[4 marks]

Hence, find the cartesian equation of the tangent plane to the above ellipsoid at the point $\mathbf{a} = (2, 1, 1)$.

[4 marks]

(b) Using the definition of gradient ($\nabla$) and curl ($\nabla\times$), show that

$$\nabla \times (\frac{\mathbf{v}}{\phi}) = \frac{\phi\nabla \times \mathbf{v} - \nabla\phi \times \mathbf{v}}{\phi^2} \, ,$$

for any (smooth enough) scalar field $\phi$ and vector field $\mathbf{v}$.

[4 marks]

Further, verify that

$$\nabla \times (\mathbf{r}) = \mathbf{0} \, , \ \nabla\left(r^3\right) = 3r\mathbf{r} \, , \ \text{and } \nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\mathbf{r} \, , r \neq 0 \, ,$$

where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $r = |\mathbf{r}|$.

[4 marks]

Deduce the expression of

$$\nabla \times \nabla\left(\frac{1}{r}\right) \, , \ r \neq 0 \, .$$

Discuss briefly the result. [4 marks]

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**2.** (a) Evaluate the line integral

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathbf{dr} \quad ,$$

where

$$\mathbf{F} = (2x + y)\hat{\mathbf{i}} + (x + 2yz^2)\hat{\mathbf{j}} + 2y^2z\hat{\mathbf{k}} \quad ,$$

and the curve $\mathcal{C}_1$ is the helix parametrised by the equations

$$x(t) = \cos t \ , \ y(t) = t \ , z(t) = \sin t \ \ , 0 \le t \le 2\pi \ .$$

[7 marks]

(b) Using the definition of gradient ($\nabla$) and curl ($\nabla\times$), show that

$$\nabla \times \nabla\phi = \mathbf{0} \ ,$$

for any (smooth enough) scalar function $\phi$.

[4 marks]

Show that the vector field $\mathbf{F}$ in (a) is irrotational, that is, $\nabla \times \mathbf{F} = \mathbf{0}$.

[5 marks]

Deduce that $\mathbf{F}$ can then be expressed as the gradient of a scalar field $\phi$, and find this scalar field.

[5 marks]

(c) Finally, evaluate the line integral

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot \mathbf{dr} \quad ,$$

where $\mathcal{C}_2$ is the straight line segment from the point $(1, 0, 0)$ to the point $(1, 2\pi, 0)$. Compare with the result of (a) and comment.

[4 marks]

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**3.** State Gauss's theorem for a differentiable vector field $\mathbf{F}$ defined over a volume $\tau$ with bounding surface $S$.

[5 marks]

(a) We want to verify this theorem for a given vector function $\mathbf{F}$ integrated over the rectangular box $S$ with corners $(\pm 1, -2, -1)$, $(\pm 1, -2, 1)$, $(\pm 1, 2-1)$ and $(\pm 1, 2, 1)$.

Calculate the divergence of the vector field

$$\mathbf{F} = x\hat{\mathbf{i}} + 2y^2\hat{\mathbf{j}} + 3z\hat{\mathbf{k}} \ .$$

[3 marks]

Then evaluate the volume integral

$$\int_\tau (\nabla \cdot \mathbf{F}) d\tau \ ,$$

where $\tau$ is the interior of $S$. [3 marks]

Finally, evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S} \ ,$$

where $C_1$ is the boundary of the surface $S_1$ above, traversed in the counterclockwise direction.

[7 marks]

(b) Apply Gauss's theorem to evaluate the surface integral

$$\int_S \mathbf{r} \cdot d\mathbf{S}$$

where $S$ is a closed surface and $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

[7 marks]

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**4.** State Stokes' theorem for a differentiable vector field $\mathbf{F}$ defined over a surface $S$ bounded by a closed curve $\mathcal{C}$.

[5 marks]

(a) We want to verify this theorem for a given vector function $\mathbf{F}$ integrated over the surface $S$ of a paraboloid

$$2z = x^2 + y^2 \ ,$$

bounded by the horizontal plane $z = 2$.

Calculate the curl of the vector field

$$\mathbf{F} = 3y\hat{\mathbf{i}} - xz\hat{\mathbf{j}} + yz^2\hat{\mathbf{k}} \ .$$

[3 marks]

Then evaluate the surface integral

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \ .$$

[6 marks]

Finally, evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \ ,$$

where $C$ is the boundary

$$x^2 + y^2 = 4 \ ,$$

of the surface $S$ above, traversed in the clockwise direction.
[Hint: you may use the parameters $x = 2\cos t$, $y = 2\sin t$, $z = 2$, where $0 \le t \le 2\pi$].

[4 marks]

(b) Apply Stokes' theorem to show that

$$\int_{\mathcal{C}} \nabla\Phi \cdot d\mathbf{r} = \mathbf{0}$$

for any smooth scalar field $\Phi$ over a closed curve $\mathcal{C}$. [7 marks]

**5.** A scalar function $V(x,y)$ obeys Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad (1)$$

in the square region $(0 \le x \le L), (0 \le y \le L)$, and is subject to the following boundary conditions

$$V(x,0) = 0 \ , V(x,L) = 0 \ , \qquad (2)$$

$$V(0,y) = 0 \ , V(L,y) = V_0 \ , \qquad (3)$$

where $V_0$ denotes a constant.

(a) Use separation of variables $V(x,y) = X(x)Y(y)$ to show that (1) decouples into

$$\frac{d^2X}{dx^2} - \alpha^2 X = 0 \ , \ \alpha \ne 0 \ , \qquad (4)$$

and

$$\frac{d^2Y}{dy^2} + \alpha^2 Y = 0 \ , \ \alpha \ne 0 \ . \qquad (5)$$

[6 marks]

From (2) and (3), deduce the boundary conditions associated with (4) and (5). Hence show that the eigenvalues of (4) and (5) are

$$\alpha = \frac{n\pi}{L} \ , \ n = 0, \ 1, \ 2, \cdots$$

and their associated eigenvectors are

$$X_n(x) = A_n \sinh(n\pi x/L) \ , Y_n(y) = D_n \sin(n\pi y/L) \ .$$

[8 marks]

(b) Show that the solution of the boundary value problem (1)-(3) can be expressed as

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1}^{\infty} \frac{\sinh((2n-1)\frac{\pi x}{L})}{(2n-1)\sinh((2n-1)\pi)} \sin((2n-1)\frac{\pi y}{L}) \ .$$

[Hint: you may assume that $\int_0^L \sin(n\pi y/L)\sin(k\pi y/L)dy = \frac{L}{2}$, if $n = k$, $n \ne 0$, and 0 otherwise.]

[9 marks]

(c) Finally, deduce that at the centre of the square, $0 \le V \le V_0$. This result holds everywhere within the square (*Maximum principle*). [Hint: you may assume that $\sinh(2s) = 2\sinh s \cosh s$ and also $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + ....$]. [2 marks]

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**6.** The displacement $V(x,t)$ from the horizontal of a uniform elastic string of unstretched length $a$ satisfies the wave equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \,, \tag{6}$$

where $c$ is a strictly positive constant (speed of wave).
This equation is subject to the boundary conditions

$$V(0,t) = V(a,t) = 0 \,, \tag{7}$$

together with the initial condition

$$V(x,0) = 0 \,, 0 < x < a \,. \tag{8}$$

(a) Show, using separation of variables $V(x,t) = X(x)T(t)$, that (6) decouples into

$$\frac{d^2 X}{dx^2} + \alpha^2 X = 0 \,,\ \alpha \neq 0 \,, \tag{9}$$

and

$$\frac{d^2 T}{dt^2} + c^2\alpha^2 T = 0 \,,\ \alpha \neq 0 \,. \tag{10}$$

[6 marks]

Deduce that the most general solution of (6)-(8) is

$$V(x,t) = \sum_{n=1}^{\infty} A_n D_n \sin(\alpha_n x) \sin(\alpha_n ct) \ ,\ \alpha_n = \frac{\pi}{a} n \,. \tag{11}$$

Find the constants $A_n D_n$ given that

$$\frac{\partial V}{\partial t}(x,0) = U \,, 0 < x < a \,, \tag{12}$$

where $U$ is a constant.

[8 marks]

(b) Show that the solution of (6)-(8) and (12) can be expressed as

$$V(x,t) = \frac{2aU}{\pi^2 c}\left[\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left((2n-1)(x-ct)\frac{\pi}{a}\right) - \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left((2n-1)(x+ct)\frac{\pi}{a}\right)\right] . \tag{13}$$

Briefly discuss this result from a physical viewpoint.
[Hint: you may assume that $\int_0^a \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi m x}{a}\right) dx = \frac{a}{2}$, if $n = m$, $n \neq 0$, and 0 otherwise.] [11 marks]