PAPER CODE NO.
MATH283

THE UNIVERSITY
of LIVERPOOL

ANSWERS TO JANUARY 2005 EXAMINATIONS

Bachelor of Engineering: Year 2
Bachelor of Science: Year 2
Master of Engineering: Year 2
Master of Physics: Year 2

FIELD THEORY AND PARTIAL DIFFERENTIAL EQUATIONS

TIME ALLOWED : Two Hours

COMMENTS

Student are asked to attempt FOUR problems only. All problems are of equal value (25 marks).
All questions were discussed either during the lectures (in which case we will call them *bookwork*) or during tutorials and homeworks (*standard exercise*).

1. (a) [*Standard exercise, similar to lectures and tutorials.*]

Given that

$$\phi(x,y,z) = x^2 + 2y^2 + 3z^2 \quad ,$$

its gradient $\nabla\phi$ is :

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (2x, 4y, 6z) \; .$$

[2 marks]

The directional derivative of the scalar function ϕ at a general point (x,y,z) in the direction given by the vector $\mathbf{b} = (1,0,0)$ writes as

$$\mathcal{D}_{\mathbf{b}}\Phi = \nabla\Phi \cdot \frac{\mathbf{b}}{|\,\mathbf{b}\,|} = (2x, 4y, 6z) \cdot \frac{(1,0,0)}{\sqrt{1}} \; .$$

[2 marks]

At the point $\mathbf{a} = (2,1,1)$, it reduces to

$$\mathcal{D}_{\mathbf{b}}\Phi(\mathbf{a}) = \nabla\Phi(\mathbf{a}) \cdot \frac{\mathbf{b}}{|\,\mathbf{b}\,|} = (4,4,6) \cdot \frac{(1,0,0)}{\sqrt{1}} = 4 \; .$$

[1 marks]

The unit *outward* normal to the ellipsoid $\phi(x,y,z) = x^2 + 2y^2 + 3z^2 = 9$ at a general point (x,y,z) is (note that the sign is positive)

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{|\,\nabla\phi\,|} = \frac{(2x,4y,6z)}{\sqrt{4x^2+16y^2+36z^2}} = \frac{(x,2y,3z)}{\sqrt{x^2+4y^2+9z^2}} \; .$$

[3 marks]

At point $\mathbf{a} = (2,1,1)$ it can be simplified as:

$$\hat{\mathbf{n}} = \frac{(2,2,3)}{\sqrt{17}} \; .$$

[1 marks]

Hence, the cartesian equation for the tangent plane which touches the ellipsoid at that point is given by

$$(\mathbf{r} - \mathbf{a}) \cdot \hat{\mathbf{n}} = 0$$
$$(\mathbf{r} - (2,2,1)) \cdot \frac{(2,2,3)}{\sqrt{17}} = 0$$
$$2(x-2) + 2(y-2) + 3(z-1) = 0$$

[3 marks]

Or

$$2x + 2y + 3z = 11 \; .$$

[1 marks]

[Altogether 13 marks]

(b) [*Bookwork i.e similar to lectures, tutorials and homeworks*].
From the definition of divergence ($\nabla\cdot$), we know that

$$\begin{aligned}\nabla\cdot(\frac{\mathbf{v}}{\phi}) &= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot\left(\phi^{-1}v_x,\phi^{-1}v_y,\phi^{-1}v_z\right)\\ &= \frac{\partial(\phi^{-1}v_x)}{\partial x}+\frac{\partial(\phi^{-1}v_y)}{\partial y}+\frac{\partial(\phi^{-1}v_z)}{\partial z}\end{aligned}$$

for any (smooth enough) scalar field $\phi(x, y, z)$ and vector field $\mathbf{v}(x, y, z) = (v_x(x, y, z), v_y(x, y, z), v_z(x, y, z))$. [2 marks]
Now,

$$\begin{aligned}\frac{\partial}{\partial x}\left(\phi^{-1}v_x\right) &= \phi^{-1}\frac{\partial}{\partial x}(v_x)+v_x\frac{\partial}{\partial x}\left(\phi^{-1}\right)\\ &= \phi^{-1}\frac{\partial}{\partial x}(v_x)-v_x\phi^{-2}\frac{\partial\phi}{\partial x}\\ &= \phi^{-2}\left(\phi\frac{\partial v_x}{\partial x}-v_x\frac{\partial\phi}{\partial x}\right)\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y}\left(\phi^{-1}v_y\right)=\phi^{-2}\left(\phi\frac{\partial v_y}{\partial y}-v_y\frac{\partial\phi}{\partial y}\right)$$

$$\frac{\partial}{\partial z}\left(\phi^{-1}v_z\right)=\phi^{-2}\left(\phi\frac{\partial v_z}{\partial z}-v_z\frac{\partial\phi}{\partial z}\right)$$

[1 marks]

Therefore,

$$\begin{aligned}\nabla\cdot\left(\frac{\mathbf{v}}{\phi}\right) &= \phi^{-2}\Bigg[\phi\left(\frac{\partial(v_x)}{\partial x}+\frac{\partial(v_y)}{\partial y}+\frac{\partial(v_z)}{\partial z}\right)\\ &\quad -\left(v_x\frac{\partial}{\partial x}(\phi)+v_y\frac{\partial}{\partial y}(\phi)+v_z\frac{\partial}{\partial z}(\phi)\right)\Bigg]\\ &= \frac{\phi\nabla\cdot\mathbf{v}-\mathbf{v}\cdot\nabla\phi}{\phi^2}\end{aligned}$$

[1 marks]

[*This is a standard exercise on gradient and divergence discussed in lectures (see chapter 3 of lecture notes) and tutorials 2 and 3.*]
We check that

$$\begin{aligned}\nabla\left(\frac{1}{r}\right) &= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-1/2}\\ &= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x,2y,2z)\\ &= -\frac{1}{r^3}\mathbf{r}\,,\ r\neq 0\,,\end{aligned}$$

[2 marks]

and

$$\begin{aligned}\nabla(r^3) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{3/2} \\ &= \frac{3}{2}(x^2+y^2+z^2)^{1/2}(2x, 2y, 2z) \\ &= 3r\mathbf{r}\ ,\end{aligned}$$

[1 marks]

as well as

$$\begin{aligned}\nabla\cdot(\mathbf{r}) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(x, y, z) \\ &= 1+1+1 \\ &= 3\ ,\end{aligned}$$

[1 marks]

where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $r = |\mathbf{r}|$.

[*Standard exercise* Therefore, we end up with

$$\begin{aligned}\nabla^2\left(\frac{1}{r}\right) &= \nabla\cdot\left(-\frac{\mathbf{r}}{r^3}\right) \\ &= -\frac{r^3\nabla\cdot\mathbf{r} - \mathbf{r}\cdot\nabla(r^3)}{r^6} \\ &= -\frac{3r^3}{r^6} + \mathbf{r}\cdot\left(\frac{3r}{r^6}\mathbf{r}\right) \\ &= -\frac{3}{r^3} + \frac{3r^3}{r^6} = 0\ ,\ r\neq 0\ .\end{aligned}$$

[4 marks]

Alternative derivation: Otherwise, one may go through this derivation without using (i) (which is more cumbersome!). In this case, we first write:

$$\begin{aligned}\nabla^2\left(\frac{1}{r}\right) &= \nabla\cdot\nabla\left(\frac{1}{r}\right) \\ &= \nabla\cdot\left(-\frac{3}{r^3}\mathbf{r}\right)\ ,\ r\neq 0\ .\end{aligned}$$

[1 marks]

Now, we note that

$$\begin{aligned}\nabla\cdot(r^{-3}\mathbf{r}) &= (\partial/\partial x, \partial/\partial y, \partial/\partial z)\cdot\frac{(x, y, z)}{r^3} \\ &= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\end{aligned}$$

$$
\begin{aligned}
&= x\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) + \frac{1}{r^3}\frac{\partial x}{\partial x} + y\frac{\partial}{\partial y}\left(\frac{1}{r^3}\right) + \frac{1}{r^3}\frac{\partial y}{\partial y} \\
&\quad + z\frac{\partial}{\partial z}\left(\frac{1}{r^3}\right) + \frac{1}{r^3}\frac{\partial z}{\partial z}\ ,\ r \neq 0\ .
\end{aligned}
$$

[1 marks]

Also,

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) &= -3r^{-4}\frac{\partial r}{\partial x} \\
&= -3r^{-4}\frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{1/2}\right) \\
&= -3r^{-4}\frac{x}{r} = -3r^{-5}x\ ,\ r \neq 0\ .
\end{aligned}
$$

[1 marks]

A similar expression holds for the other two variables y and z. We thus find that

$$
\begin{aligned}
\nabla \cdot (r^{-1}\mathbf{r}) &= x^2\left(-\frac{1}{r^3}\right) + \frac{1}{r^3} + y^2\left(-\frac{1}{r^3}\right) + \frac{1}{r^3} + z^2\left(-\frac{1}{r^3}\right) + \frac{1}{r^3} \\
&= \frac{3}{r^3} - \frac{3r^2}{r^5} = 0\ r \neq 0\ .
\end{aligned}
$$

[1 marks]

In physics, $1/r$ is known as Coulomb potential which is a standard solution of Laplace equation $\nabla^2\phi = 0$.

[Altogether 12 marks]

Adding up (a) and (b) we obtain a total of **25 marks** as required.

2. (a) [*Standard exercise, similar to lectures and tutorials.*]

We want to evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr} \,,$$

where

$$\mathbf{F} = (y+z)\hat{\mathbf{i}} + (z+x)\hat{\mathbf{j}} + (x+y)\hat{\mathbf{k}} \,,$$

and the curve $\mathcal{C}$ is defined by

$$y = x^2 \,, z = x^3 \,,$$

from the point $(0,0,0)$ to $(1,1,1)$.

For this, we first parametrise the path $\mathcal{C}$ in 3D space by the equations

$$\begin{aligned} x &= t \\ y &= t^2 \\ z &= t^3 \,, \end{aligned}$$

where t starts at 0 and ends at 1. [1 marks]

We then need to express the vector field $\mathbf{F}$ in terms of the parameter t

$$\mathbf{F}(t) = (t^2+t^3)\hat{\mathbf{i}} + (t^3+t)\hat{\mathbf{j}} + (t+t^2)\hat{\mathbf{k}} \,.$$

[1 marks]

Further, if we consider the parametrised vector

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + t^3\hat{\mathbf{k}} \,,$$

its total derivative writes as

$$\frac{d\mathbf{r}}{dt} = \hat{\mathbf{i}} + 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}} \,.$$

[1 marks]

Altogether, the line integral of the vector field $\mathbf{F}$ over the path $\mathcal{C}$ is expressed as

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr} &= \int_0^1 \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \\ &= \int_0^1 \left(t^2+t^3, t^3+t, t+t^2\right) \cdot \left(1, 2t, 3t^2\right) \, dt \\ &= \int_0^1 \left(t^2+t^3+2t^4+2t^2+3t^3+3t^4\right) \, dt \\ &= \int_0^1 \left(5t^4+4t^3+3t^2\right) \, dt \\ &= \Big[t^5+t^4+t^3\Big]_0^1 \\ &= 1+1+1 = 3 \,. \end{aligned}$$

[4 marks]

[Altogether 7 marks]

(b) From the definition of gradient (∇) and curl ($\nabla\times$), we have

$$\begin{aligned}\nabla \times \nabla\phi &= \left(\frac{\partial^2}{\partial y\partial z}\phi - \frac{\partial^2}{\partial z\partial y}\phi\right)\hat{\mathbf{i}} \\ &- \left(\frac{\partial^2}{\partial x\partial z}\phi - \frac{\partial^2}{\partial z\partial x}\phi\right)\hat{\mathbf{j}} \\ &+ \left(\frac{\partial^2}{\partial x\partial y}\phi - \frac{\partial^2}{\partial y\partial x}\phi\right)\hat{\mathbf{k}} .\end{aligned}$$

[3 marks]

Now, for any smooth enough scalar function ϕ, Schwarz's theorem ensures that [*Bookwork: theorem proved in lecture notes for functions of class C^2.*]

$$\frac{\partial^2}{\partial y\partial z} = \frac{\partial^2}{\partial z\partial y} ,$$

so that the term sitting as a factor in front of $\hat{\mathbf{i}}$ cancels out. Similarly, the other two factors sitting respectively in front of $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ also cancel out. Hence, we conclude that for any (smooth enough) scalar field ϕ

$$\nabla \times \nabla\phi = \mathbf{0} .$$

[1 marks]

[*Throughout the M283 course, we assumed that the domain over which the functions were defined were simply connected, so that the students did not meet any advanced theoretical problems when dealing with a curl-free gradient (cohomology).*] [altogether 4 marks]

[*Standard exercise addressed both in lectures and tutorial 2.*] The curl of the vector field

$$\mathbf{F} = (y+z)\hat{\mathbf{i}} + (z+x)\hat{\mathbf{j}} + (x+y)\hat{\mathbf{k}} ,$$

is given by

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right)\hat{\mathbf{i}} \\ &- \left(\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right)\hat{\mathbf{j}} \\ &+ \left(\frac{\partial}{\partial y}(y+z) - \frac{\partial}{\partial x}(z+x)\right)\hat{\mathbf{k}} \\ &= (1-1)\hat{\mathbf{i}} - (1-1)\hat{\mathbf{j}} + (1-1)\hat{\mathbf{k}} = \mathbf{0} .\end{aligned}$$

[3 marks for definition of curl and 2 marks for result i.e. 5 marks] [*standard exercise detailed both in lectures and tutorials*] We deduce that $\mathbf{F}$ can be expressed as the gradient of a scalar field ϕ, since on the one hand it is irrotational and on the other hand the gradient of any scalar field is itself irrotational. Hence,

$$\mathbf{F} = (y+z)\hat{\mathbf{i}} + (z+x)\hat{\mathbf{j}} + (x+y)\hat{\mathbf{k}} = \nabla\phi\ .$$

[1 marks]

In other words,

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= y+z \\ \frac{\partial\phi}{\partial x} &= z+x \\ \frac{\partial\phi}{\partial x} &= x+y\ , \end{aligned}$$

[2 marks]

so that

$$\begin{aligned} \phi &= yx + zx + f_1(x,y) \\ \phi &= zy + xy + f_2(x,z) \\ \phi &= xz + yz + f_3(x,y)\ , \end{aligned}$$

where f_1, f_2 and f_3 are three arbitrary functions which we need specify.

By inspection, we end up with

$$\phi = yx + zx + zy + C$$

[2 marks]

where C is an arbitrary constant which can be set to zero.

[Altogether 5 marks]

[*Bookwork: this exercise has to do with the Gradient theorem and conservative fields and was addressed both in tutorials and lectures as well as set in homeworks.*] The line integral in (a) can be reformulated as

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \nabla\phi\cdot d\mathbf{r} \\ &= \int_0^1 \nabla\phi\cdot\frac{d\mathbf{r}}{dt}dt \\ &= \int_0^1 \left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right)\cdot\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)dt \\ &= \int_0^1 \frac{d\phi}{dt}dt \\ &= \phi(1,1,1) - \phi(0,0,0) \\ &= 1+1+1 = 3\ , \end{aligned}$$

[3 marks]

which is consistent with the result of question **2**.(a).

[*This derivation is fairly classical and was introduced during the lectures within the context of so-called 'Gradient theorem'. Mentioning this theorem without going through the above derivation also provides a full mark.*]

[1 marks]

[Altogether for (b) 18 marks]

If we add up marks for (a) and (b) we find **25 marks** as required.

3. [*Bookwork, similar exercises were set in lectures, tutorials and homeworks.*

Let us state Gauss's theorem for a differentiable vector field $\mathbf{F}$ defined over a volume τ with bounding surface S.

Gauss's (or divergence's) theorem :

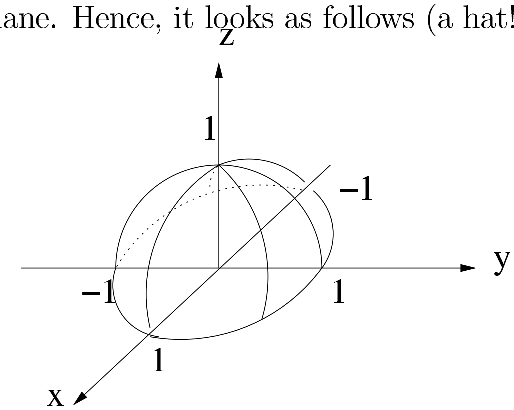
Given a volume τ which is bounded by a piecewise continuous surface S, and a vector function $\mathbf{F}$ which is continuous and has continuous partial derivatives in τ, then

$$\iiint_\tau \nabla \cdot \mathbf{F} d\tau = \oint_S \mathbf{F} \cdot d\mathbf{S} \quad . \tag{1}$$

Notice that the surface integral S has been drawn with a circle around it, in order to indicate that this surface is *closed*, i.e. that it entirely encompasses the volume τ. Also, $d\mathbf{S}$ is defined as the product of a small area (let's say $dxdy$) by the unit outward normal $\hat{\mathbf{n}}$ to the surface S. [5 marks]

The exercise now splits in two sections (which can be independently addressed).

(a) The region τ is the hemi-spherical volume which is enclosed by the
surface $x^2 + y^2 + z^2 = 1$ and the plane $z = 0$, and lies above the (x, y) plane. Hence, it looks as follows (a hat!):



[2 marks]

Using Gauss's theorem, the surface integral can be expressed as

$$\begin{aligned}
\oint_S (x\hat{\mathbf{i}} + 4y\hat{\mathbf{j}} + 7z\hat{\mathbf{k}}) \cdot d\mathbf{S} &= \iiint_\tau \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x, 4y, 7z)\ d\tau \\
&= \iiint_\tau (1 + 4 + 7)\, d\tau \\
&= 12\,(\text{Volume of region } \tau) \\
&= 12 \frac{4}{3} \frac{\pi 1^3}{2} = 8\pi \ .
\end{aligned}$$

[3 marks]

(b) Using polar coordinates,

$$x = r\sin\theta\ ,\ y = r\cos\theta\ ,$$

the region D can be reformulated as

$$\begin{aligned} D(r,\theta) &= \left\{(r,\theta)\ :\ 2r^2\cos^2\theta + r^2\sin^2\theta \le 8\ ,\ r\ge 0\ ,\ 0\le\theta\le\pi/2\right\} \\ &= \left\{(r,\theta)\ :\ 0\le r\le R(\theta) = \frac{2\sqrt{2}}{\sqrt{2\cos^2\theta+\sin^2\theta}}\ ,0\le\theta\le\pi/2\right\} \end{aligned}$$

[2 marks]

So that we can express the double integral as

$$\begin{aligned} \iint_D f(x,y)\ dxdy &= \int_0^{\pi/2}\int_0^{R(\theta)} f(r,\theta)\ dr(rd\theta) \\ &= \int_0^{\pi/2}\left(\int_0^{R(\theta)} r^3dr\right)\sin\theta\cos\theta\ d\theta \\ &= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{R(\theta)}\sin\theta\cos\theta\ d\theta \\ &= \int_0^{\pi/2}\left(\frac{2\sqrt{2}}{\sqrt{2\cos^2\theta+\sin^2\theta}}\right)^4\sin\theta\cos\theta\ d\theta \end{aligned}$$

[5 marks]

As suggested by the "Hint", the student should now make the change of variable $u = \cos^2\theta$. Of course, we now have $d\theta = -1/2\sin\theta\cos\theta d\theta$, so that

$$\begin{aligned} \iint_D f(x,y)\ dxdy &= \frac{1}{2}\int_0^1 \frac{64}{(2u+1-u)^2}\ du \\ &= 32\left[-\frac{1}{u+1}\right]_0^1 \\ &= 16 \end{aligned}$$

[3 marks]

The student should now make use of Gauss's theorem which ensures that

$$\begin{aligned} \iint_{\partial(D\times[0,1])}(x^2y\hat{\mathbf{i}})\cdot d\mathbf{S} &= \iiint_{D\times[0,1]}\left(\nabla\cdot(x^2y\hat{\mathbf{i}})\right)\ dxdydz \\ &= \iiint_{D\times[0,1]}(2xy)\ dxdydz \\ &= 2\int_0^1\left(\iint_D xy\ dxdy\right)\ dz \\ &= 2\times 16 \\ &= 32 \end{aligned}$$

[3 marks for Gauss's theorem+2 marks for result=5 marks]

Adding up **5 marks** for stating Gauss's theorem, **5 marks** for (a), **10 marks** for the double integral in (b) and **5 marks** for the surface integral in (b), we end up with **25 marks** as required.

4. [*bookwork (same spirit as exercise* **3***).*]

Let us first state the Stokes' theorem for a differentiable vector field **F** defined over a surface S bounded by a closed curve $\mathcal{C}$.
Stokes' theorem:
Given a surface S which is bounded by a piecewise continuous curve $\mathcal{C}$, and a vector function **F** which is continuous and has continuous partial derivatives on S, then

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \tag{2}$$

[5 marks]

The exercise now splits into two sections.

(a) The curl of the vector field

$$\mathbf{F_1} = xz^2\hat{\mathbf{i}} + 2x\hat{\mathbf{j}} + zx^2\hat{\mathbf{k}} \ .$$

writes as

$$\begin{aligned} \nabla \times \mathbf{F_1} &= (0-0)\,\hat{\mathbf{i}} - (2xz - 2zx)\,\hat{\mathbf{j}} + \hat{\mathbf{k}}\,(2-0) \\ &= 2\hat{\mathbf{k}} \ . \end{aligned}$$

[3 marks]

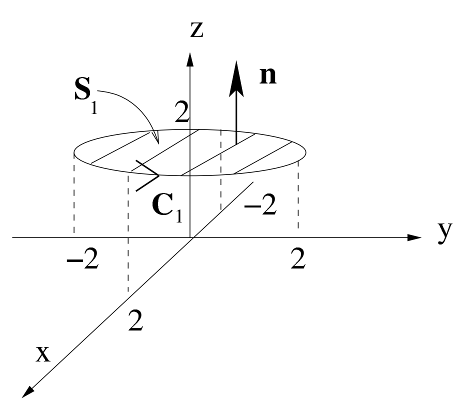
Let us now evaluate the surface integral

$$\int\int_{S_1} (\nabla \times \mathbf{F_1}) \cdot d\mathbf{S}$$

where S_1 is the plane surface bounded by the circular path

$$x^2 + y^2 = 4 \quad , \qquad z = 2 \quad .$$

which is depicted on the figure below so that the unit normal $\hat{\mathbf{n}}$ to the surface $\mathbf{S_1}$ is simply the vector of the canonical basis $\hat{\mathbf{k}}$ (we deduce the orientation of $\hat{\mathbf{n}}$ from the thumb rule).



Thus, an infinitesimal element $d\mathbf{S}$ of orientable surface $\mathbf{S_1}$ writes as $d\mathbf{S} = \hat{\mathbf{n}}dxdy = \hat{\mathbf{k}}dxdy$ and we end up with

$$\begin{aligned}\int\int_{S_1} (\nabla \times \mathbf{F_1}) \cdot d\mathbf{S} &= \int\int_{S_1} (2\hat{\mathbf{k}}) \cdot \hat{\mathbf{k}} dxdy = \int\int_S 2dxdy \\ &= 2\,(\text{area of } S_1) \\ &= 2\left(\pi 2^2\right) = 8\pi \;.\end{aligned}$$

[3 marks]

Let us now evaluate the path integral

$$\int_{\mathcal{C}_1} \mathbf{F_1} \cdot d\mathbf{r} \;,$$

where $\mathcal{C}_1$ is the boundary of the surface S_1 above, traversed in the counterclockwise direction.
First of all we use the parametrisation

$$\begin{aligned} x &= 2\cos t \\ y &= 2\sin t \\ z &= 2 \end{aligned}$$

with

$$0 \leq t \leq 2\pi \quad .$$

Then

$$\mathbf{F_1}(t) = 8\cos t\hat{\mathbf{i}} + 4\cos t\hat{\mathbf{j}} + 8\cos^2 t\hat{\mathbf{k}} \quad ,$$

[1 marks]

also

$$\mathbf{r}(t) = 2\cos t\hat{\mathbf{i}} + 2\sin t\hat{\mathbf{j}} + 1\hat{\mathbf{k}}$$

hence

$$\frac{d\mathbf{r}}{dt} = -2\sin t\hat{\mathbf{i}} + 2\cos t\hat{\mathbf{j}} + 0\hat{\mathbf{k}} \quad .$$

[1 marks]

Therefore, the integral is

$$\begin{aligned}\oint_{\mathcal{C}_1} \mathbf{F_1} \cdot d\mathbf{r} &= 8\int_0^{2\pi} \left(-2\cos t\sin t + \cos^2 t\right) dt \\ &= 8\left(-\int_0^{2\pi} \sin(2t)dt + \int_0^{2\pi} \frac{1+\cos(2t)}{2} dt\right) \\ &= 8\left[\frac{\cos(2t)}{2}\right]_0^{2\pi} + \left[\frac{t}{2} + \frac{\sin(2t)}{2}\right]_0^{2\pi} \\ &= 8\pi \;.\end{aligned}$$

[2 marks]

Hence, we have checked that

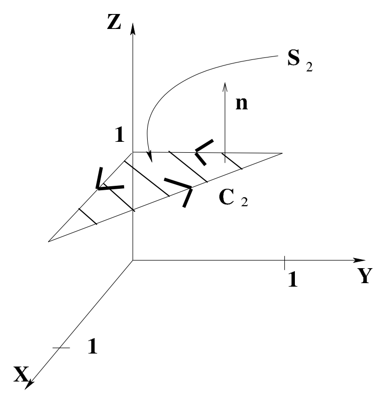
$$\int\int_{S_1} (\nabla \times \mathbf{F_1}) \cdot d\mathbf{S} = 8\pi = \oint_{C_1} \mathbf{F_1} \cdot d\mathbf{r} \ . \tag{3}$$

[Altogether for (a) 10 marks]

(b) We start with the surface integral. We note that

$$\begin{aligned} \nabla \times \mathbf{F_2} &= \hat{\mathbf{i}}(0-0) - \hat{\mathbf{j}}(0-z^2) + \hat{\mathbf{k}}(0-x) \\ &= z^2\hat{\mathbf{j}} - x\hat{\mathbf{k}} \\ &= z^2\hat{\mathbf{j}} - x\hat{\mathbf{k}} \ \text{ on } S_2. \end{aligned}$$

[3 marks]



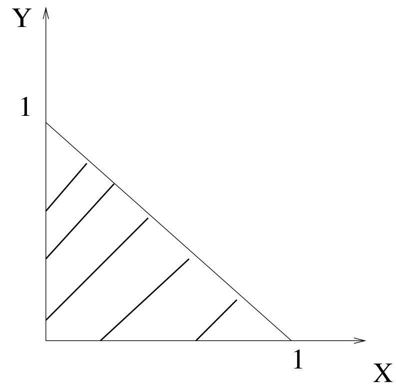
Also on S_2 we have

$$d\mathbf{S} = \hat{\mathbf{k}} dx \ dy \quad .$$

So the surface integral is

$$\begin{aligned} \int\int_{S_2} \nabla \times \mathbf{F_2} \cdot d\mathbf{S_2} &= \int\int_{S_2} (z^2\hat{\mathbf{j}} - x\hat{\mathbf{k}}) \cdot \hat{\mathbf{k}} dx \ dy \\ &= \int_{S_2} dx \ dy(-x) \\ &= \int_0^1 dx \int_0^{1-x} dy(-x) \\ &= \int_0^1 dx(-x) \left[y\right]_0^{1-x} \\ &= \int_0^1 dx(-x+x^2) \\ &= -\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = -\frac{1}{6} \end{aligned}$$

[3 marks]



Now looking at the line integral, we split it up into three sections:

$$\oint_{\mathcal{C}_2} \mathbf{F_2} \cdot d\mathbf{r} = \int_{\mathcal{C}_2^1} \mathbf{F_2} \cdot d\mathbf{r} + \int_{\mathcal{C}_2^2} \mathbf{F_2} \cdot d\mathbf{r} + \int_{\mathcal{C}_2^3} \mathbf{F_2} \cdot d\mathbf{r}$$

On the line segment $\mathcal{C}_2^1$, we have $z = 1, x = 0$ and y starts at 1 and ends at 0. Hence we choose the parameters:

$$\begin{aligned} y &= t \\ x &= 0 \\ z &= 1 \end{aligned} \tag{4}$$

Hence

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= \hat{\mathbf{i}}\frac{dx}{dt} + \hat{\mathbf{j}}\frac{dy}{dt} + \hat{\mathbf{k}}\frac{dz}{dt} \\ &= \hat{\mathbf{j}} \end{aligned} \tag{5}$$

So,

$$\begin{aligned} \mathbf{F_2} \cdot \frac{d\mathbf{r}}{dt} &= (y^3\hat{\mathbf{j}}) \cdot \hat{\mathbf{j}} \\ &= t^3 \ . \end{aligned} \tag{6}$$

Hence

$$\begin{aligned} \int_{\mathcal{C}_2^1} \mathbf{F_2} \cdot \frac{d\mathbf{r}}{dt} dt &= \int_1^0 t^3 dt \\ &= \left[\frac{t^4}{4}\right]_1^0 = -\frac{1}{4} \end{aligned}$$

[1 marks]

On the line segment $\mathcal{C}_2^2$, we have $z = 1, y = 0$ and x varies between 0 and 1. Hence we chhose $x = t$ as our parameter, giving

$$\frac{d\mathbf{r}}{dt} = \hat{\mathbf{i}}$$

and hence

$$\mathbf{F_2} \cdot \frac{d\mathbf{r}}{dt} = t\hat{\mathbf{k}} \cdot \hat{\mathbf{i}} \cdot = 0 \ .$$

which implies

$$\int_{\mathcal{C}_2^2} \mathbf{F_2} \cdot \frac{d\mathbf{r}}{dt} dt = 0 \quad .$$

Therefore this line segment contributes nothing to the final integral. [1 marks]

On the line segment $\mathcal{C}_2^3$, we have $z = 1$, $x + y = 1$ and x starts at $x = 1$ and finishes at $x = 0$. Hence we chose the parameters

$$\begin{aligned} z &= 1 \\ x &= t \\ y &= 1 - t \end{aligned} \tag{7}$$

Thus we have

$$\frac{d\mathbf{r}}{dt} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$$

which implies

$$\begin{aligned} \mathbf{F_2} \cdot \frac{d\mathbf{r}}{dt} &= (xy\hat{\mathbf{i}} + y^3\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} - \hat{\mathbf{j}}) \\ &= t(1-t) - (1-t)^3 \end{aligned} \tag{8}$$

So that the integral over the line segment $\mathcal{C}_2^3$ is

$$\begin{aligned} \int_{\mathcal{C}_2^3} \mathbf{F_2} \cdot \frac{d\mathbf{r}}{dt} dt &= \int_1^0 \Big((t - t^2) - (1-t)^3 \Big) dt \\ &= \left[\frac{t^2}{2} - \frac{t^3}{3} + \frac{(1-t)^4}{4} \right]_1^0 \\ &= \frac{1}{4} - (\frac{1}{2} - \frac{1}{3}) = \frac{1}{12} \end{aligned}$$

[1 marks]

The final integral is then

$$\begin{aligned} \int_{\mathcal{C}_2} \mathbf{F_2} \cdot d\mathbf{r} &= -\frac{1}{4} + 0 + \frac{1}{12} \\ &= -\frac{1}{6} \end{aligned} \tag{9}$$

And so Stokes theorem is verified. [1 marks]

[Altogether for (b) 10 marks]

Adding up **5 marks** for the statement of Stokes's theorem, **10 marks** for (a) and **10 marks** for (b) we obtain a total of **25 marks as required**.

5. [*Standard exercise, the Laplace equation has been addressed in class work and also in lecture notes.*]

(a) The function we want to find satisfies Laplace's equation in two dimensions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{10}$$

in the region $0 \le x \le \pi$, $-L \le y \le L$, and the boundary condition which we prescribe on the edge of this region are

$$V(-L, y) = V(L, y) = 1 \quad , \tag{11}$$
$$V(x, 0) = V(x, \pi) = 0 \quad . \tag{12}$$

We try a solution which has the form

$$V(x, y) = X(x)Y(y) \quad , \tag{13}$$

and now we have to work out what these functions $X(x)$ and $Y(y)$ are. Substituting into Laplace's equation, we obtain

$$X''(x)Y(y) + Y''(y)X(x) = 0 \tag{14}$$

which we can rearrange to get

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0 \quad . \tag{15}$$

[2 marks]

The point is that the variables separate, and we see that we have to solve the two ordinary differential equations

$$\frac{d^2X}{dx^2} = \pm\alpha^2 X(x) \quad , \frac{d^2Y}{dy^2} = \mp\alpha^2 Y(y) \quad . \tag{16}$$

[2 marks]

To get the right choice for the sign of α^2 we look at the boundary conditions; the second condition (12) implies that

$$Y(0) = Y(\pi) = 0 \quad , \tag{17}$$

and so this function cannot end up being made up of cosh and sinh functions otherwise it would be zero, and so the differential equations we have to solve are

$$\frac{d^2X}{dx^2} = +\alpha^2 X(x) \quad , \tag{18}$$
$$\frac{d^2Y}{dy^2} = -\alpha^2 Y(y) \quad . \tag{19}$$

[2 marks]

[Altogether 6 marks]

The general solution to the differential equation (19) is

$$Y(y) = C\cos(\alpha y) + D\sin(\alpha y) \tag{20}$$

[2 marks]

The boundary condition $Y(0) = 0$ lets us get rid of one of the arbitrary constants:

$$0 = C\cos(\alpha 0) + D\sin(\alpha 0)$$

and so $C = 0$. Applying the boundary condition $Y(\pi) = 0$ gives us a restraint on α:

$$0 = D\sin(\alpha\pi)\ ,$$

[2 marks]

so that we are left with

$$\alpha := \alpha_n = n \qquad \text{where } n = 0, 1, 2, \ldots \tag{21}$$

[1 marks]

We now have the eigenvectors of the differential equation (19):

$$Y_n(y) = D_n \sin(ny) \quad \text{, where } n = 0, 1, 2, \ldots \tag{22}$$

[1 marks]

We now turn attempt to solve the first differential equation (18). The general solution to this equation is

$$X(x) = A\cosh(\alpha x) + B\sinh(\alpha x) \tag{23}$$

[2 marks]

Now, looking at the boundary conditions for our function $X(x)$, we notice that from (11) these are

$$X(-L) = X(L) = 1\ . \tag{24}$$

[1 marks]

These boundary conditions are inherently *symmetric* in the variable x. This means that we can eliminate (anti-symmetric) $\sinh \alpha x$ terms from the general solution (23). Hence, we can write the eigenvectors of (18) as

$$X_n(x) = A_n \cosh(\alpha_n x)\ . \tag{25}$$

[1 marks]

[Altogether 10 marks]

(b) Combining the functions of x and y, we obtain

$$\begin{aligned} V_n(x,y) &= X_n(x)Y(y) \\ &= A\cosh(\alpha_n x)D_n\sin(ny) \\ &= E_n\cosh(nx)\sin(ny)\ , \end{aligned} \tag{26}$$

since we already know $\alpha_n = n$ from (21). [1 marks]

We now use the principle of superposition, which says that we can add any two solutions together to get another solution, to write the general solution (our differential operator is linear)

$$V(x,y) = \sum_{n=0}^{\infty} E_n\cosh(nx)\sin(ny) \quad . \tag{27}$$

[1 marks]

Now if we pick the coefficients E_n correctly then we can satisfy the final boundary condition, which is

$$V(L,y) = 1\ .$$

(Notice that if this boundary condition is satisfied, then it automatically follows that $V(-L,y) = 1$ because we have already imposed symmetry on the solution). So we want to pick coefficients E_n such that

$$1 = \sum_{n=0}^{\infty} E_n\cosh(nL)\sin(ny)\ . \tag{28}$$

[1 marks]

In order to work out the unknown coefficients E_n we have to exploit the orthogonality of the functions $\sin(ny)$. This means that the functions are linearly independent, that is, is is impossible to make up, say, the function $\sin 3y$ out of any linear cominbation of $\sin y$, $\sin 2y$, $\sin 4y$, etc. More practically it means that multiples of orthogonal functions always integrate to zero over certain regions. In this case

$$\int_0^{\pi} \sin(ny)\sin(ky)dy = \begin{cases} \frac{\pi}{2} & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

[1 marks]

We now multiply equation (28) by the function $\sin(ky)$, and integrate it between 0 and π:

$$\begin{aligned} \int_0^{\pi} \sin(ky)dy &= \sum_{n=1}^{\infty} E_n\cosh(nL)\int_0^{\pi} \sin(ny)\sin(ky)dy \\ &= E_k\cosh(2kL)\frac{\pi}{2}\ , \end{aligned} \tag{30}$$

[1 marks]

where we have used the orthogonality relation (29) to eliminate all terms but one from the infinite sum.

The integral on the left hand side is easily calculated to be:

$$\begin{aligned} \int_0^\pi \sin(ky)dy &= \left[-\frac{\cos(ky)}{k}\right]_0^\pi \\ &= -\frac{\cos(k\pi)-1}{k} \ . \end{aligned} \tag{31}$$

[1 marks]

Combining these gives

$$-\frac{\cos(k\pi)-1}{k} = E_k \cosh(kL)\frac{\pi}{2} \ , \tag{32}$$

[1 marks]

Or, re-arranging,

$$E_k = \frac{2}{\pi k}\frac{1-\cos(k\pi)}{\cosh(kL)} \ . \tag{33}$$

[1 marks]

We can now write down the full solution to the differential equation:

$$\begin{aligned} V(x,y) &= \sum_{n=1}^{\infty} E_n \cosh(nx)\sin(ny) \\ &= \sum_{n=0}^{\infty} \frac{2}{\pi}\frac{1-\cosh(n\pi)}{n\cosh(nL)}\cosh(nx)\sin(ny) \ . \end{aligned} \tag{34}$$

[1 marks]

[Altogether 9 marks]

Adding up (a) and (b) we have **25 marks** as required.

6. [*Standard exercise: the heat equation has been addressed during classwork and also in lecture notes.*]

(a) To solve the equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \tag{35}$$

we try a solution of the form

$$u(x,t) = X(x)T(t) \tag{36}$$

Substituting (36) into (35) leads to

$$X(x)T'(t) = \kappa X''(x)T(t) \ , \tag{37}$$

which can be recast as

$$\frac{X''(x)}{X(x)} = \frac{1}{\kappa}\frac{T'(t)}{T(t)} \ , \tag{38}$$

[3 marks]

[Here, we have used the fact that X(x) and T(t) are $\neq 0$, since we are looking for non-trivial solutions of the spectral problems in X and T.]

Hence, we must solve

$$\frac{X''(x)}{X(x)} = -\alpha^2 \ , \ \frac{1}{\kappa}\frac{T'(t)}{T(t)} = -\alpha^2 \ . \tag{39}$$

[*Anticipating the exponential solution in $T(t)$, we have picked a negative separation constant $-\alpha^2$, so that the solution remains finite when t tends to infinity. This was explained during class exercise and in the lecture notes.*] [1 marks]

To find the eigenvalues and eigenfunctions of the boundary value problem

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0 \ , \quad X(0) = X(\pi) = 0 \ \ , \tag{40}$$

we proceed as follows:

We first notice that the general solution to:

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0 \ ,$$

is

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x) \ .$$

[2 marks]

When $x = 0$, this simplifies to:

$$X(0) = A\cos(\alpha 0) + B\sin(\alpha 0) = A$$

which thanks to boundary condition $X(0) = 0$ implies

$$A = 0\ .$$

[1 marks]

Now, looking at the other extremity of the bar ($X(l) = 0$) gives:

$$B\sin(\alpha l) = 0\ .$$

[1 marks]

So that we are left with:

$$\alpha := \alpha_n l = \pi n\ ,\ n = 0,\ 1,\ 2, \ldots$$

[2 marks]

Thus, the eigenvalues of the spectral problem (40) look like:

$$\alpha_n = \frac{\pi}{l} n\ ,\ n = 0,\ 1,\ 2, \ldots \tag{41}$$

and the associated eigenfunctions can be written as:

$$X_n(x) = B_n \sin(\alpha_n x)\ .$$

[1 marks]

The eigenfunctions associated with the first ordinary differential equation in (39) subject to boundary conditions $X(0) = X(l) = 0$ are

$$X_n(x) = B_n \sin(\frac{\pi}{l} n x)$$

[1 marks]

The second equation in (39) is

$$T'(t) = -\alpha^2 \kappa T(t)\ ,$$

[1 marks]

which has the general solution

$$T_n(t) = C_n \exp(-\alpha_n^2 \kappa t)\ .$$

[1 marks]

[Altogether 14 marks]

(b) We can write the general solution $u(x,t)$ as

$$\begin{aligned} u(x,t) &= \sum_0^\infty X_n(x)T_n(t) \\ &= \sum_0^\infty B_n \sin\left(\frac{\pi}{l}nx\right) C_n \exp(-\alpha_n^2 \kappa t) \\ &= \sum_0^\infty \Bigg[B_n \sin\left(\frac{\pi}{l}nx\right) \\ &\quad \times C_n \exp\left(-\kappa \frac{(\pi n)^2}{l^2} \kappa t\right)\Bigg] , \end{aligned} \tag{42}$$

where in the last equation we have used (41). [3 marks]

We know that at $t = 0$

$$u(x,0) = x(l-x) \; .$$

We thus deduce from (42) that

$$x(l-x) = \sum_0^\infty B_n\, C_n \sin\left(\frac{\pi}{l}nx\right) \; .$$

Let us now multiply both sides of this equation by $\sin(\pi m x/l)$

$$x(l-x)\sin\left(\frac{\pi}{l}mx\right) = \sum_0^\infty B_n\, C_n \sin\left(\frac{\pi}{l}nx\right) \sin\left(\frac{\pi}{l}mx\right) \; ,$$

and integrate over the interval $[0,l]$

$$\begin{aligned} \int_0^l x(l-x)\sin\left(\frac{\pi}{l}mx\right)\, dx &= \sum_0^\infty B_n\, C_n \int_0^l \Bigg[\sin\left(\frac{\pi}{l}nx\right) \\ &\quad \sin\left(\frac{\pi}{l}mx\right)\, dx\Bigg] \; . \end{aligned} \tag{43}$$

[3 marks]

Now, we note that

$$\int_0^l \sin\left(\frac{\pi}{l}nx\right) \sin\left(\frac{\pi}{l}mx\right)\, dx = \begin{cases} \frac{l}{2} & \text{for} \quad m = n, \\ 0 & \text{otherwise,} \end{cases} \tag{44}$$

so that

$$B_n\, C_n \frac{l}{2} = \int_0^l x(l-x)\sin\left(\frac{\pi}{l}mx\right)\, dx \; ,$$

where $m = n$. [1 marks]

Integrating by parts twice, we obtain:

$$\begin{aligned}
B_n\, C_n\, \frac{l}{2} &= \left[\frac{-\cos\left(\frac{\pi m x}{l}\right)}{\frac{\pi m}{l}} x(1-x)\right]_0^l - \int_0^l \left[-\frac{\cos\left(\frac{\pi m x}{l}\right)}{\frac{\pi m}{l}}\right](1-2x)\, dx \\
&= \int_0^l \frac{\cos\left(\frac{\pi m x}{l}\right)}{\frac{\pi m}{l}}(1-2x)\, dx \\
&= \left[\frac{\sin\left(\frac{\pi m x}{l}\right)}{\frac{\pi^2 m^2}{l^2}}(1-2x)\right]_0^l - \int_0^l \frac{\sin\left(\frac{\pi m x}{l}\right)}{\frac{\pi^2 m^2}{l^2}}(-2)\, dx \\
&= 2\left(\frac{l}{m\pi}\right)^2 \int_0^l \sin\left(\frac{\pi m x}{l}\right)\, dx \\
&= 2\left(\frac{l}{m\pi}\right)^2 \left[\frac{-\cos\left(\frac{\pi m x}{l}\right)}{\frac{\pi m}{l}}\right]_0^l \\
&= 2\left(\frac{l}{m\pi}\right)^3 \Big(1-\cos(\pi m)\Big)\, . \qquad (45)
\end{aligned}$$

From (45), it is clear that

$$B_n\, C_n\, \frac{l}{2} = \begin{cases} 4\left(\dfrac{l}{m\pi}\right)^3 & \text{for} \quad m = 2k+1, \quad k = 0, 1, 2\cdots \\ 0 & \text{otherwise,} \end{cases} \qquad (46)$$

Therefore, since $m = n$, it follows from (42) and (46) that the solution of the heat equation (35) subject to the boundary conditions $u(0,t) = u(l,t) = 0$ and the initial condition $u(x,0) = x(l-x)$, writes as:

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} B_n\, C_n \sin\left(\frac{\pi}{l} n x\right) \exp\left(\frac{-(\pi n)^2 \kappa t}{l^2}\right) \\
&= \sum_{k=0}^{\infty} B_{2k+1}\, C_{2k+1} \sin\left(\frac{\pi}{l}(2k+1)x\right) \exp\left(\frac{-(\pi(2k+1))^2 \kappa t}{l^2}\right) \\
&= \frac{8l^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}\left[\sin\left(\frac{(2n+1)\pi x}{l}\right)\right. \\
&\qquad \left. \times \exp\left(\frac{-(2n+1)^2 \pi^2 \kappa t}{l^2}\right)\right]. \qquad (47)
\end{aligned}$$

[1 marks]

[Altogether 11 marks]

Adding up (a) and (b) we obtain **25 marks** as required.