PAPER CODE NO.
MATH283

THE UNIVERSITY *of* LIVERPOOL

JANUARY 2005 EXAMINATIONS

Bachelor of Engineering: Year 2
Bachelor of Science: Year 2
Master of Engineering: Year 2
Master of Physics: Year 2

FIELD THEORY AND PARTIAL DIFFERENTIAL EQUATIONS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Attempt FOUR questions only. All questions are of equal value (25 marks each).

Throughout the paper $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ represent unit vectors parallel to the x, y and z axes respectively.

1. (a) Given that

$$\phi(x,y,z) = x^2 + 2y^2 + 3z^2 \quad ,$$

calculate $\nabla\phi$.

[2 marks]

Derive the expression for the directional derivative $\mathcal{D}_b\phi(\mathbf{a})$ of ϕ at point $\mathbf{a} = (2,1,1)$ in the direction of the vector $\mathbf{b} = (1,0,0)$.

[3 marks]

Calculate the outward unit normal to the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 9$$

at the point $\mathbf{a} = (2,1,1)$.

[4 marks]

Hence, find the cartesian equation of the tangent plane to the above ellipsoid at the point $\mathbf{a} = (2,1,1)$.

[4 marks]

(b) Using the definition of gradient (∇) and divergence ($\nabla\cdot$), show that

$$\nabla \cdot (\frac{\mathbf{v}}{\phi}) = \frac{\phi\nabla\cdot\mathbf{v} - \nabla\phi\cdot\mathbf{v}}{\phi^2} \, ,$$

for any (smooth enough) scalar field ϕ and vector field $\mathbf{v}$.

[4 marks]

Further, verify that

$$\nabla\cdot(\mathbf{r}) = 3 \, , \; \nabla\left(r^3\right) = 3r\mathbf{r} \, , \text{ and } \nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\mathbf{r} \, , r \neq 0 \, ,$$

where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $r = |\mathbf{r}|$.

[4 marks]

Deduce the expression of

$$\nabla^2\left(\frac{1}{r}\right) \, , \; r \neq 0 \, ,$$

where $\nabla^2 = \nabla\cdot\nabla$ denotes the Laplacian.

[4 marks]

2. (a) Evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr} \quad ,$$

where

$$\mathbf{F} = (y+z)\hat{\mathbf{i}} + (z+x)\hat{\mathbf{j}} + (x+y)\hat{\mathbf{k}} \, ,$$

the curve $\mathcal{C}$ joins the origin to the point $(1, 1, 1)$, and it is defined by

$$y = x^2 \, , z = x^3 \, .$$

[7 marks]

(b) Using the definition of gradient (∇) and curl ($\nabla\times$), show that

$$\nabla \times \nabla\phi = \mathbf{0} \, ,$$

for any (smooth enough) scalar function ϕ.

[4 marks]

Show that the vector field

$$\mathbf{F} = (y+z)\hat{\mathbf{i}} + (z+x)\hat{\mathbf{j}} + (x+y)\hat{\mathbf{k}}$$

is irrotational, that is, $\nabla \times \mathbf{F} = \mathbf{0}$.

[5 marks]

Deduce that $\mathbf{F}$ can then be expressed as the gradient of a scalar field ϕ, and find this scalar field.

[5 marks]

Finally, verify that the line integral in (a) can be evaluated as

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr} = \phi(1, 1, 1) - \phi(0, 0, 0) \, .$$

[4 marks]

3. State Gauss's theorem for a differentiable vector field $\mathbf{F}$ defined over a volume τ with bounding surface S.

[5 marks]

We want to apply this theorem to evaluate two surface integrals.

(a) If the region τ is the hemi-spherical volume defined as

$$\tau = \left\{ (x, y, z) : x^2 + y^2 + z^2 \leq 1 \; , \; z \geq 0 \right\} ,$$

sketch τ and evaluate the surface integral

$$\int\int_{S_1} (x\hat{\mathbf{i}} + 4y\hat{\mathbf{j}} + 7z\hat{\mathbf{k}}) \cdot d\mathbf{S}$$

where S_1 is the bounding surface of τ. [5 marks]

(b) Use polar coordinates to find the double integral of the scalar function $f(x, y) = xy$ over the the quarter ellipse

$$D = \left\{ (x, y) : x^2 + \frac{y^2}{2} \leq 4 \; , \; x \geq 0 \; , \; y \geq 0 \right\} .$$

[Hint: You may find the change of variable $u = \cos^2 \theta$ helpful.]

[10 marks]

Hence, evaluate the surface integral

$$\int\int_{S_2} (x^2 y\hat{\mathbf{i}}) \cdot d\mathbf{S}$$

where S_2 is the surface of the cylindrical solid $D \times [0, 1]$.

[5 marks]

4. State Stokes' theorem for a differentiable vector field **F** defined over a surface S bounded by a closed curve $\mathcal{C}$.

[5 marks]

We want to verify this theorem in two cases.

(a) Calculate the curl of the vector field

$$\mathbf{F_1} = xz^2\hat{\mathbf{i}} + 2x\hat{\mathbf{j}} + zx^2\hat{\mathbf{k}} \ .$$

[3 marks]

Then evaluate the surface integral

$$\int\int_{S_1} (\nabla \times \mathbf{F_1}) \cdot d\mathbf{S} \ ,$$

where S_1 is the upward-oriented plane surface in $z = 2$ bounded by the circular path

$$x^2 + y^2 = 4 \ .$$

[3 marks]

Finally, evaluate the line integral

$$\int_{C_1} \mathbf{F_1} \cdot d\mathbf{r} \ ,$$

where C_1 is the boundary of the surface S_1 above, traversed in the counterclockwise direction. [4 marks]

(b) Calculate the curl of the vector field

$$\mathbf{F_2} = xy\hat{\mathbf{i}} + y^3\hat{\mathbf{j}} + xz^2\hat{\mathbf{k}} \ .$$

[3 marks]

Then evaluate the surface integral

$$\int\int_{S_2} (\nabla \times \mathbf{F_2}) \cdot d\mathbf{S} \ ,$$

where S_2 is the triangle with corners at points $(0, 0, 1)$, $(1, 0, 1)$ and $(0, 1, 1)$. [3 marks]

Finally, evaluate the line integral

$$\int_{C_2} \mathbf{F_2} \cdot d\mathbf{r} \ ,$$

where C_2 is the boundary of the surface S_2 above, traversed in the counterclockwise direction. [4 marks]

5. A scalar function $V(x,y)$ obeys Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{1}$$

in the rectangular region $(-L \leq x \leq L), (-\pi/2 \leq y \leq \pi/2)$, and is subject to the following boundary conditions

$$V(x,0) = V(x,\pi) = 0 \ , \tag{2}$$
$$V(-L,y) = V(L,y) = 1 \ . \tag{3}$$

(a) Use separation of variables $V(x,y) = X(x)Y(y)$ to show that (1) decouples into

$$\frac{d^2X}{dx^2} - \alpha^2 X = 0 \ , \ \alpha \neq 0 \ , \tag{4}$$

and

$$\frac{d^2Y}{dy^2} + \alpha^2 Y = 0 \ , \ \alpha \neq 0 \ . \tag{5}$$

[6 marks]

From (2) and (3), deduce the boundary conditions associated with (4) and (5). Hence show that the eigenvalues of (4) and (5) are

$$\alpha = n \ , \ n = 0, \ 1, \ 2, \cdots$$

and their associated eigenvectors are

$$X_n(x) = A_n \cosh(nx) \ , Y_n(y) = D_n \sin(ny) \ .$$

[10 marks]

(b) Finally, show that the solution of the boundary value problem (1)-(3) can be expressed as

$$V(x,y) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1-\cos(n\pi)}{n} \frac{\cosh(nx)}{\cosh(nL)} \sin(ny) \ .$$

[Hint: you may assume that $\int_0^\pi \sin(ny)\sin(ky)dy = \frac{\pi}{2}$, if $n = k$, and 0 otherwise.]
[9 marks]

6. The flow of heat in a thin bar of length l is governed by the heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \tag{6}$$

where κ is a strictly positive constant (thermal diffusivity).
This equation is subject to the boundary conditions

$$u(0,t) = u(l,t) = 0 \,,$$

together with the initial condition

$$u(x,0) = x(l-x) \,.$$

(a) Show, using separation of variables $u(x,t) = X(x)T(t)$, that (6) decouples into

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0 \,,\; \alpha \neq 0 \,, \tag{7}$$

and

$$\frac{dT}{dt} + \kappa\alpha^2 T = 0 \,,\; \alpha \neq 0 \,. \tag{8}$$

[4 marks]

Hence show that the eigenvalues of (7) and (8) are

$$\alpha_n = \frac{\pi}{l} n \,,\; n = 0,\ 1,\ 2, \cdots$$

and their associated eigenvectors are

$$X_n(x) = B_n \sin(\alpha_n x) \,, T_n(t) = C_n \exp(-\alpha_n^2 \kappa t) \,.$$

[10 marks]

(b) Finally, show that the solution of the heat equation (6) can be expressed here as

$$u(x,t) = \frac{8l^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin\left(\frac{(2n+1)\pi x}{l}\right) \exp\left(\frac{-(2n+1)^2 \pi^2 \kappa t}{l^2}\right) \,.$$

[Hint: you may assume that $\int_0^l \sin\left(\frac{\pi n x}{l}\right) \sin\left(\frac{\pi m x}{l}\right) dx = \frac{l}{2}$, if $n = m$, and 0 otherwise. You may also integrate twice by parts to show that $\int_0^l x\,(l-x) \sin\left(\frac{\pi m x}{l}\right) dx = \frac{4l^3}{(m\pi)^3}$ if m is odd, and 0 otherwise.]

[11 marks]