PAPER CODE NO.
MATH283

THE UNIVERSITY
of LIVERPOOL

JANUARY 2001 EXAMINATIONS

Bachelor of Engineering: Year 2
Bachelor of Science: Year 2
Master of Engineering: Year 2
Master of Physics: Year 2

FIELD THEORY AND PARTIAL DIFFERENTIAL EQUATIONS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Attempt FOUR questions only. All questions are of equal value.

In this paper $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ represent unit vectors parallel to the x, y and z axes respectively.

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1. (a) Given that

$$\phi(x, y, z) = x^2 + y^2 + 2z^2 \quad ,$$

find $\nabla\phi$. Hence calculate the outward pointing unit normal to the ellipsoid

$$x^2 + y^2 + 2z^2 = 10$$

at the point $(2, 2, 1)$. Use this to find the cartesian equation of the tangent plane at this point.

[13 marks]

(b) Calculate the divergence of the vector function

$$\mathbf{v} = \frac{\mathbf{r}}{r^3} \quad ,$$

where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $r = |\mathbf{r}|$. [12 marks]

2. (a) Evaluate the path integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr} \quad ,$$

where

$$\mathbf{F} = (2x + yz)\hat{\mathbf{i}} + (2y + xz)\hat{\mathbf{j}} + (2z + xy)\hat{\mathbf{k}}$$

and the curve $\mathcal{C}$ is the straight line starting at the point $(1, 1, 1)$ and finishing at $(0, 0, 0)$. [10 marks]

(b) State Gauss's theorem for a differentiable vector field $\mathbf{F}$ defined over a volume τ with bounding surface S.

[5 marks]

The region τ is the hemi-spherical volume which is enclosed by the
surface $x^2 + y^2 + z^2 = 1$ and the plane $z = 0$, and lies above the (x, y) plane.

Sketch τ and, using Gauss's theorem or otherwise, evaluate the surface integral

$$\int\int_S (x\hat{\mathbf{i}} + 4y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot d\mathbf{S}$$

where S is the bounding surface of τ. [10 marks]

3. State Stokes' theorem for a differentiable vector field $\mathbf{F}$ over a surface S bounded by a closed curve $\mathcal{C}$.

[5 marks]

Calculate the curl of the vector field

$$\mathbf{F} = yz^2\hat{\mathbf{i}} + 2xz\hat{\mathbf{j}} + xy^2\hat{\mathbf{k}} \ .$$

[5 marks]

Evaluate the surface integral

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where S is the plane surface bounded by the circular path

$$x^2 + y^2 = 1 \quad , \qquad z = 1 \quad .$$

[7 marks]

Evaluate the path integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \ ,$$

where C is the boundary of the surface S above, traversed in the anti-clockwise direction. (Hint: you may use the parameters $x = \cos t$, $y = \sin t$, $z = 1$, where $0 \leq t \leq 2\pi$). Hence verify Stokes' theorem in this case.

[8 marks]

4. Show that the general solution to the differential equation

$$\frac{d^2X}{dx^2} - \alpha^2 X = 0$$

is, when $\alpha \neq 0$,

$$X = A\cosh(\alpha x) + B\sinh(\alpha x) \quad .$$

[5 marks]

A scalar function $V(x, y)$ obeys Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

in the region $(0 \leq x \leq \pi), (0 \leq y \leq \pi)$, and is subject to the boundary conditions

$$V(x, 0) = V(x, \pi) = V(0, y) = 0 \ ,$$
$$V(\pi, y) = \frac{1}{2}\sin(4y) \ .$$

Use separation of variables to show that

$$V(x, y) = \frac{1}{2}\frac{1}{\sinh(4\pi)}\sinh(4x)\sin(4y) \ .$$

[20 marks]

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5. Find the eigenvalues and eigenfunctions of the boundary value problem

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0 \ , \quad X(0) = X(\pi) = 0 \ .$$

[10 marks]

The flow of heat in a thin bar of length π is governed by the heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

and is subject to the boundary conditions $u(0,t) = u(\pi,t) = 0$, together with the inital condition

$$u(x,0) = \begin{cases} x, & 0 \le x \le \pi/2 \\ x - \pi, & \pi/2 < x \le \pi \ . \end{cases}$$

Show, using separation of variables, that

$$u(x,t) = -\sum_{n=1}^{\infty} \frac{2}{n} \cos\left(\frac{n\pi}{2}\right) \sin\left(nx\right) e^{-n^2\kappa t} \ .$$

[15 marks]