THE UNIVERSITY *of* LIVERPOOL

1. An investor has £10,000 capital. He can leave the money in a bank (with annual profit of 4%), or buy stock. He estimates the chances that stock appreciates as 60%; in this case he gains £1,000 in the coming year. Otherwise, he loses £1,000.

Alternatively, the investor can consult a broker. The broker is not 100 % reliable. If the stock is in fact going to appreciate, then with probability 0.8 his advice will be to invest, if the stock is going to depreciate, then with probability 0.8 his advice will be not to invest. If the investor consults the broker he has to pay a fee of £C.

(a) Draw the decision tree for the investor. [4 marks]

(b) Calculate all the probabilities for the chance nodes. [7 marks]

(c) Elaborate the optimal policy for the investor (in terms of C). [7 marks]

(d) What is the maximal value of C that the investor should agree to pay to the broker? [2 marks]

2. An automatic machine produces A (thousands of) units of a product per day. As A increases, the proportion of defectives, q, goes up according to the following probability density function

$$f(q) = \begin{cases} Aq^{A-1}, & \text{if } 0 \le q \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate the mathematical expectation $E[q]$ and hence the expected amount of defective items $A \cdot E[q]$, in terms of A. [5 marks]

(b) Suppose each defective item incurs a loss of £50; a good item yields £5 profit. Give an expression for the expected total daily net profit, P, in terms of A. [2 marks]

Suppose also the production of one thousand units results in 0.5 kg of unwanted smoke and analyse the bicriterion problem:

$$\begin{aligned} P &= E[\text{total daily net profit}] \to \max \\ S &= \text{total daily amount of smoke} \to \min, \end{aligned}$$

that is

(c) give an expression for $P(S)$ and indicate the range of S corrresponding to the non-inferior solutions; [8 marks]

(d) sketch the graph of the trade-off curve. [5 marks]

3. A shop manager can order 100, 150, 200, 250, or 300 loaves of bread. Assume for simplicity that the demand is also restricted to one of the 5 levels specified above. The manager pays 25 pence a loaf and sells them for 60 pence each. Any loaves that are not sold by the end of the day are disposed of at 10 pence each.

(a) Write down the matrix of rewards. [2 marks]

(b) Find the optimal strategy using the $MaxMin$ criterion. [4 marks]

(c) Find the optimal strategy using Savage's criterion of opportunity loss. [8 marks]

(d) Suppose demand is specified by the following probability distribution:

Demand:	100	150	200	250	300
p_n:	.20	.25	.30	.15	.10

Find the optimal strategy using the expected reward criterion. [6 marks]

4. The FTSE100 share prices x_n, $n = 1, 2, \ldots, 6$, in 6 consecutive trading days were

Day	1	2	3	4	5	6
Price	4852.30	4906.20	4916.20	4908.30	4941.50	4979.80

(a) Starting from $t = 2$, perform the moving-average forecasting procedure with $n = 2$, i.e. calculate F_{t+1}, $t = 2, 3, 4, 5$. [4 marks]
Calculate MAD, the Mean Absolute Deviation. [1 marks]

(b) Do the same for $n = 3$, $t = 3, 4, 5$ and calculate MAD. [4 marks]

(c) Starting from $t = 2$, perform Holt's forecasting procedure with $\alpha = 0.6$ and $\beta = 0.7$, the level-related and the slope-related smoothing constants. To put it differently, compute the forecasts F_{t+1}, $t = 2, 3, 4, 5$. [8 marks]
(You can take the second observation x_2 as the initial value of the level S_2, and the difference $x_2 - x_1$ as the initial value of the slope B_2.)

– calculate MAD. [1 marks]

(d) Choose the method providing the minimal value of MAD and use it to predict the next FTSE100 price, x_7. [2 marks]

5. A wholesale distributor of bicycles wishes to forecast sales of a particular model of bicycle. The seasonal model without linear trend was agreed to be suitable. The seasonal sales figures for both last year and this year (to date) are shown in the table:

Season	Last year, 2005	This year, 2006
Winter	2,786	2,800
Spring	2,928	2,925
Summer	3,025	3,040
Autumn	3,061	3,035

(a) Take the average seasonal sales in 2005 as the initial level S_0 and evaluate the initial values of the seasonal indices, $L_{-3}, L_{-2}, L_{-1}, L_0$ as the ratios of actual sales in 2005 to S_0. Explain why $p = 4$ is the appropriate value for the period. [5 marks]

(b) Encode Winter-2006 as 1, Spring-2006 as 2 and so on. Neglect the linear trend, i.e. put $B_0 = 0$ and $\beta = 0$, the slope related smoothing constant. Take $\alpha = 0.1$ and $\gamma = 0.2$ (level and season related smoothing constants) and apply Winter's method to year 2006. [12 marks]

(c) Calculate the forecast for Winter 2007. [3 marks]

6. A supermarket has two girls ringing up sales at the counters. The service time for each customer is exponential with mean 4 minutes; on average, 10 customers arrive every hour, the inter-arrival times being again exponentially distributed.

(a) Describe all possible states of this queuing system and draw the transition diagram. [3 marks]

(b) Write down equations for the steady-state probabilities of the states. [6 marks]

(c) What is the steady-state probability of having no customers in the system? [6 marks]

(d) What is the steady-state probability of having to wait for service? [5 marks]

7. (a) For the Pareto density function

$$f(t) = \begin{cases} 0, & t < a; \\ \beta a^{\beta} t^{-\beta-1}, & t \geq a, \end{cases}$$

where $a > 0,\ \beta > 1$, derive expressions for the mathematical expectation and the cumulative distribution function $F(t)$. [4 marks]

Using the inverse transformation method, show that a Pareto random variable (RV) T can be calculated by the formula

$$T = \frac{a}{U^{1/\beta}}, \qquad (*)$$

where $U \sim U(0,1)$ is standard uniform. [3 marks]

(b) Put $a = 1$ and $\beta = 3$. Calculate the mean for this Pareto RV. Having the following 4 realizations of the standard uniform RV u_i

0.68 0.91 0.05 0.33,

calculate corresponding values T_i^A using (*). [2 marks]

(c) Suppose RV $T = T^A$ given above describes the interarrival times in a one-channel queuing system with infinite waiting space. Assume that the service time T^S is Uniform(0,C) with $C = 2$. What is the mean service time? Is it reasonable to expect that the equilibrium exists? [4 marks]

(d) Having the following 4 realizations of the standard uniform RV v_i

0.91 0.27 0.39 0.65,

calculate corresponding values T_i^S using the (obvious) formula: $T_i^S = 2 \times v_i$. [2 marks]

(e) Let $n(t)$ be the number of requests in the system, which is empty initially. Draw the simulated graph of $n(t)$ based on T_i^A and T_i^S calculated earlier. Assume there were no other requests during time interval [0,10]. [5 marks]