1. A machine is repaired as soon as it breaks down. On day T, preventive maintenance is performed, and the machine becomes as good as new. Let $p_1, p_2, \ldots$ be the probabilities that the new machine would break down in the first day, second day, and so on. Assume that C_1 is the cost of repairing a broken machine and C_2 is the preventive maintenance cost. The total cost per day $C(T)$, under a fixed value of $T \geq 1$, equals

$$C(T) = \frac{C_1 \sum_{t=1}^{T-1} X_t + C_2}{T},$$

where $X_t = \begin{cases} 1 & \text{if the new machine gets broken in day } t; \\ 0 & \text{otherwise} \end{cases}$
is a Bernoulli random variable, $P(X_t = 1) = p_t$.

(a) Write down the expression for the expected cost per day $E[C(T)]$. [3 marks]

(b) Write down the expressions for $Var[X_t]$ and hence for $Var[C(T)]$ assuming that X_t, $t = 1, 2, \ldots$ are independent random variables. [7 marks]

(c) Let $C_1 = £90$, $C_2 = £10$, $p_1 = 0.02$, $p_2 = 0.10$. Calculate the optimum T that minimizes the expected value – variance criterion

$$J = E[C(T)] + K\ Var[C(T)] \to \min$$

among the first whole numbers $T = 1, 2, 3$, under $K = 0.1$. [10 marks]

2. A company has the options now of building a full-size plant or a small plant that can be expanded two years later. The total planning period is 10 years.

In the case of a full-size plant, the revenue depends on the demand: if demand is high, the full-size plant will yield £1,000,000 annually; if demand is low, the full-size plant will yield only £300,000 annually. A market survey indicates that the probability of having high demand is 0.75.

In the case of a small plant, this probability for the next two years is obviously the same, but the small plant will yield £200,000 annually if demand is low, and £250,000 annually in the case of high demand. If demand is low, the manager will definitely not expand the plant, but if demand is high he/she must make a decision about the expansion, taking into account that demand in the remaining 8 years can again be low or high with the same probabilities of 0.25 and 0.75, independently of anything else.

The immediate construction of a large plant will cost £5 million and a small plant will cost only £1 million. The expansion of the small plant 2 years from now is estimated to cost £4.2 million. Assume there is no inflation.

(a) Draw the decision tree. [8 marks]

(b) Elaborate the optimal policy analysing the tree in the standard way. [12 marks]

3. If you invest £M in security A then, after a one year period, you get £$M(1+r_A)$, where return $r_A \sim N(\mu_A, \sigma_A^2)$. The same concerns another security B, $r_B \sim N(\mu_B, \sigma_B^2)$; $\mu_A, \mu_B > 0$; $\sigma_A, \sigma_B > 0$. Random variables r_A and r_B are independent.

Suppose you split your capital of £10000 and invest $x \cdot 10000$ in A and $(1-x) \cdot 10000$ in B ($x \in [0,1]$). If Δ is the (random) increment of the total capital then $r = \frac{\Delta}{10000}$ is the resulting return.

(a) Write down the expression for r in terms of x, r_A and r_B. [2 marks]

(b) In what follows, take $\mu_A = 0.08$, $\mu_B = 0.10$, $\sigma_A = 0.01$, $\sigma_B = 0.02$. Calculate the mean $\mu = E[r]$ and the variance $\sigma^2 = Var[r]$ in terms of x. [5 marks]

(c) What value of $x \in [0,1]$ provides the minimal value of σ^2 ? [5 marks]

(d) Investors always want to maximize μ and to minimize σ^2. Construct the tradeoff curve for this bicriteria problem. [8 marks]

4. Suppose that shares of a particular company are trading on 1 Jan. 2006 at £9.00. Assume that every two months, the price moves either 25% up or 25% down. Consider a European call option expiring on 1 May 2006, the exercise price being £8.00. Assume that the annual risk free interest rate is 6%.

(a) Draw the binomial tree of possible price movements. [3 marks]

(b) Suppose on 1 March 2006 the shares price moves down. Calculate the option price on this date. [7 marks]

(c) Calculate the initial option price on 1 Jan. 2006. [10 marks]

5. A wholesale distributor of bicycles wishes to forecast sales of a particular model of bicycle. The seasonal model without linear trend was agreed to be suitable. The seasonal sales figures for both last year and this year (to date) are shown in the table:

Season	Last year, 2004	This year, 2005
Winter	2,516	2,663
Spring	2,881	2,677
Summer	3,101	3,020
Autumn	3,048	2,987

(a) Take the average seasonal sales in 2004 as the initial level S_0 and evaluate the initial values of the seasonal indices, $L_{-3}, L_{-2}, L_{-1}, L_0$ as the ratios of actual sales in 2004 to S_0. Explain why $p = 4$ is the appropriate value for the period. [5 marks]

(b) Encode Winter-2005 as 1, Spring-2005 as 2 and so on. Neglect the linear trend, i.e. put $B_0 = 0$ and $\beta = 0$, the slope related smoothing constant. Take $\alpha = 0.1$ and $\gamma = 0.2$ (level and season related smoothing constants) and apply the Winter's method to year 2005. [12 marks]

(c) Calculate the forecast for Winter 2006. [3 marks]

6. A supermarket has three cashiers and unbounded space for waiting in one common line. The service time for each customer is exponential with mean 4 minutes; on average 10 customers arrive every hour, the inter-arrival times being again exponentially distributed.

(a) Describe all possible states of this queuing system and draw the transition diagram. [3 marks]

(b) Evaluate the parameters of the system and write down equations for the steady-state probabilities of the states. [6 marks]

(c) What is the steady-state probability of having no customers in the system? [6 marks]

(d) What is the steady-state probability of having to wait for service? [5 marks]

7. (a) For the Rayleigh density function

$$f(t) = 2te^{-t^2}, \qquad t \geq 0,$$

derive the cumulative distribution function $F(t)$. [4 marks]

(b) Show that this Rayleigh random variable T can be calculated by the formula

$$T = G(U) = \sqrt{-\ln(U)}, \qquad (*)$$

where $U \sim Uniform(0,1)$ is standard uniform random variable. [5 marks]

(c) Having the following 5 realizations of the uniform RV u_i

$$0.48 \quad 0.37 \quad 0.68 \quad 0.75 \quad 0.47,$$

calculate the corresponding values using (*) and denote them as T_i^A. [2 marks]

(d) Having the following 5 realizations of the uniform RV v_i

$$0.90 \quad 0.19 \quad 0.25 \quad 0.28 \quad 0.67,$$

calculate the corresponding values of the exponential random variable using the standard formula

$$T = -\ln v$$

and denote them as T_i^S. [2 marks]

(e) Let T_i^A be all the simulated interarrival times during the period $[0,5]$ in a one-channel queuing system with refusals (no space for waiting), which is empty initially. Let T_i^S be the service time of the i-th customer provided he is not rejected. Draw the simulated graph of $n(t)$, the number of requests in the system, $t \in [0,5]$. [7 marks]