THE UNIVERSITY
of LIVERPOOL

SECTION A

1.

Consider the following real-valued function

$$f(x) = \ln\left(\frac{1+x}{1-x}\right) + 1.$$

(i) Compute $f'(x)$ and verify that $f'(x) > 0$ in $(-1, 1)$. How many roots can $f(x)$ have in $(-1, 1)$? [4 marks]

(ii) Use 2 steps of the Newton-Raphson method to find an approximate solution to $f(x) = 0$, starting from $x^{(0)} = 0$.
(Keep at least 4 decimal digits in calculations.) [4 marks]

2. [7 marks]

Starting from $x^{(0)} = 0$ and $x^{(1)} = -0.5$, solve the nonlinear equation

$$\ln\left(\frac{1+x}{1-x}\right) + 1 = 0,$$

using 2 steps of the Secant method. Compute the error in your approximation, given the exact solution is $x = (1-e)/(1+e)$. (Keep at least 4 decimal digits in calculations.)

3. [9 marks]

Sketch the following two simultaneous nonlinear equations in x_1, x_2,

$$\begin{cases} x_1^2 + x_2^2 = 4, \\ 9(x_1 - 2)^2 + x_2^2 = 1. \end{cases}$$

Taking the initial guess $\mathbf{x}^{(0)} = (2, 1)^T$, use 1 step of the Newton-Raphson method to find an approximate solution.

4.

Consider the linear system $Ax = b$ with

$$A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 0 & 3 \\ 3 & 1 & 2 \end{pmatrix}, b = \begin{pmatrix} 6 \\ 1 \\ 4 \end{pmatrix}.$$

(i) Compute $\|A\|_\infty$ and $\|b\|_1$. [3 marks]

(ii) Solve it by Gaussian elimination. [5 marks]

(iii) Hence or otherwise find the LDM decomposition of A. [5 marks]

5. [6 marks]

For the following matrix A

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 12 & 3 \\ 1 & 0 & -10 \end{pmatrix},$$

use the Gerschgorin theorem to decide which of the following three values

$$-2, -11, 50$$

is closest to an eigenvalue of A.

6.

(i) Using the composite Trapezium rule (with 2 subintervals) approximate

$$\int_0^1 \cos(x+y)u(y)dy.$$

[4 marks]

(ii) Using this approximation for the integral, set up the linear system to find the numerical solution of the following equation

$$7u(x) - \int_0^1 \cos(x+y)u(y)dy = 2x + 1 \quad x \in [0, 1].$$

Do not solve the system. [8 marks]

SECTION B

7. [15 marks]

Use the shifted inverse power method for 2 steps to estimate both the eigenvalue near $\gamma = 6$ and its corresponding eigenvector for the following matrix

$$A = \begin{pmatrix} 9 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -14 & 8 \end{pmatrix}.$$

Start the iteration from $\mathbf{x}^{(0)} = [-1, 0, 1]^T$ and keep at least 4 decimal places in your calculations.

You may use the result

$$(A - 6I)^{-1} = \begin{pmatrix} 0.4 & 0.56 & -0.2 \\ 0 & -0.2 & 0 \\ -0.2 & -1.68 & 0.6 \end{pmatrix}.$$

8.

(a) State both the explicit and implicit Euler methods for the following general initial value problem.

$$\frac{dy}{dx} = f(x, y(x)), \quad y(0) = y_0$$

Use the explicit Euler method to solve the initial value problem

$$\frac{dy}{dx} = y^2 - 1 + ye^{-x}, \quad y(0) = 0$$

to obtain y(0.5) with the step length $h = 0.25$. [8 marks]

(b) Combine the nonlinear Newton-Raphson method with the implicit Euler method to solve

$$\frac{dy}{dx} = ye^{-5x} - y^2, \quad y(0) = 1$$

to obtain y(0.2) with the step length $h = 0.1$. (Use no more than 2 iterations in each Newton-Raphson step, and keep at least 4 decimal places in your calculations.) [7 marks]

9. [15 marks]

Consider the solution of the following boundary value problem,

$$-\left((1+x)\frac{\partial^2 u}{\partial x^2}+(1+y)\frac{\partial^2 u}{\partial y^2}\right)=(x+2y+1),\quad p=(x,y)\in\Omega$$

where the domain is the square $\Omega=[0,0.3]\times[-0.2,0.1]\in R^2$, with the Dirichlet boundary condition $u=0$ on all boundary points, to be solved by the usual finite difference (FD) method with 3×3 boxes, i.e. 4 interior and uniformly distributed mesh points. Verify that the FD equation at the point (x_1,y_1)=$(0.1,-0.1)$ is

$$-1.1u_{21}-4u_{11}-0.9u_{12}=0.009.$$

Find the remaining 3 FD equations and the final linear system (there is no need to *solve* the system).

10.

Consider the linear system $Ax=b$ where

$$A=\begin{pmatrix}9 & 2 & 0\\ 2 & 8 & -3\\ 0 & -3 & 7\end{pmatrix},b=\begin{pmatrix}13\\ 15\\ 1\end{pmatrix}.$$

(i) Write out three equations for the 3 components of the vector $x^{(n+1)}$ for the Jacobi Iteration Method and carry out 2 iterations starting from $x^{(0)}=\mathbf{0}$. Keep 2 decimal places in your calculations. Find the iteration matrix T_J and the vector c_J such that

$$x^{(n+1)}=T_Jx^{(n)}+c_J.$$

[7 marks]

(ii) Find $(L+D)^{-1}$, where L and D are the lower diagonal and the diagonal parts of A respectively. Hence compute the iteration matrix T_{GS} and the vector c_{GS} such that

$$x^{(n+1)}=T_{GS}x^{(n)}+c_{GS}.$$

[7 marks]

(iii) What necessary and sufficient conditions can you use to check whether each of the Jacobi and Gauss-Seidel Iteration Methods converge or not?

[1 marks]

11.

Design a three point quadrature of the Gauss type

$$\int_{-1}^{1} f(x)dx = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2)$$

by choosing suitable weights w_j where $x_0 = -1,\ x_1 = 0,\ x_2 = 1$. (Hint: the rule should be exact for degree $0, 1, 2$ polynomials.)

[5 marks]

Use the rule you obtained to approximate the integral

$$I = \int_{-1}^{1} \frac{\cos x dx}{\sqrt{x^3+5}}.$$

[3 marks]

Modify the rule to approximate the integral

$$I = \int_{0}^{1} \frac{\cos x dx}{\sqrt{x^3+5}}.$$

(Keep at least 4 decimal places in all calculations.) [7 marks]