SAMPLE SOLUTIONS M266 [May 2005]

F = Formula   m = marks

Section A (Easier Q's — H/W)

E = [illegible]ation/Computation   A = Answer

Q1 At points $x = 2, \frac{5}{2}$, test

$$f(2) = 0.7769 > 0$$
$$f(2.5) = -0.20067 < 0$$

$\therefore$ $[2, 2.5)$ contains one root of $f(x)$.

m4 = E3+

Next differentiate to get

$$f'(x) = -2 + \frac{1}{x+2} - \frac{1}{x+3}$$

$$= -\frac{2(x+5/2)^2 - 3/2}{(x+2)(x+3)}.$$

m2

With $x^{(0)}$ given, the NR takes the form

$$x^{(k+1)} = x^{(k)} - f(x^{(k)})/f'(x^{(k)})$$

Given $x^{(0)} = 2.5$

$$\downarrow$$

$$x^{(1)} = 2.39760$$

$$\downarrow$$

$$x^{(2)} = 2.39755$$

m6

F2 + A2 + A2

Q2 $\begin{pmatrix} -3 & -6 & 12 \\ -2 & -1 & 2 \\ 6 & 13 & -21 \end{pmatrix} \xrightarrow[R_3 = R_3 + 2R_1]{R_2 = R_2 - \frac{2}{3}R_1} \begin{pmatrix} -3 & -6 & 12 \\ 0 & 3 & -6 \\ 0 & 1 & 3 \end{pmatrix}$

$\xrightarrow{R_3 = R_3 - \frac{1}{3}R_2} \begin{pmatrix} -3 & -6 & 12 \\ 0 & 3 & -6 \\ 0 & 0 & 5 \end{pmatrix} = U.$ $\frac{m5}{=A4+A1}$

Q3 The intersection is estimated as $\frac{m2}{}$
(-1.6, 2.1) which is closer to (-2, 3).

Now consider the NR ~~method~~

$$x^{(k+1)} = x^{(k)} - J^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$ $\frac{m5}{=F_2+F_3}$

$$= x^{(k)} - \begin{pmatrix} 3x_1^{(k)^2} & 2x_2^{(k)} \\ 4x_1^{(k)^3} & 2x_2^{(k)} \end{pmatrix}^{-1} \begin{pmatrix} x_1^{(k)^3} + x_2^{(k)^2} - 2 \\ x_1^{(k)^4} + x_2^{(k)^2} - 9 \end{pmatrix}$$

With $x^{(0)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$J^{-1} = \begin{pmatrix} 3 & 4 \\ -4 & 4 \end{pmatrix}^{-1} = \frac{1}{28}\begin{pmatrix} 4 & -4 \\ 4 & 3 \end{pmatrix}$$ $\frac{m5}{E_2+F_2+A}$

$$x^{(1)} = x^{(0)} - J^{-1}\begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} -1.7143 \\ 2.2857 \end{pmatrix}.$$

Q4 The GS method for $x^{n} \to x^{n+1}$ is

$$\begin{cases} 3x_1^{(n+1)} = 20 - x_2^{(n)} \\ 4x_2^{(n+1)} = 19 - x_1^{(n+1)} - 2x_3^{(n)} \\ 4x_3^{(n+1)} = 29 - 2x_2^{(n+1)} + x_4^{(n)} \\ 3x_4^{(n+1)} = -5 + x_3^{(n+1)}, \quad n = 0, 1, 2, \dots \end{cases}$$

m3

Iterate with $x^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \\ x_4^{(0)} \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ 9 \\ 0 \end{pmatrix}$

m5

$A_3 + 1$

$$x^{(1)} = \begin{pmatrix} 6.6667 \\ -1.4167 \\ 7.9583 \\ 0.9861 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} 7.1389 \\ -1.0139 \\ 8.0035 \\ 1.0012 \end{pmatrix}$$

Q5 Obtain the Gerschgorin disks as

$$\begin{cases} |\lambda_1 - 15| \le 1 \\ |\lambda_2 - 2| \le 1 \\ |\lambda_3 + 8| \le 2 \end{cases}$$

m5

Since all disks are separated, $\lambda(A)$ are real and the appropriate choice may be

$$\begin{cases} \gamma_1 = 15 \\ \gamma_2 = 2 \\ \gamma_3 = -18 \end{cases}$$ for the e'values by IPM.

$= F + A1 + A2$

Q6 The Lagrange polynomial is

$$P_3(x) = \sum_{i=1}^{4} y_i \prod_{\substack{j=1 \\ j\neq i}}^{4} \frac{x-x_j}{x_i-x_j}$$ F2+A3 ↑ m5

$$= -\frac{(x-2)(x-4)(x-5)}{6} + \frac{4(x-1)(x-4)(x-5)}{3} - \frac{2(x-1)(x-2)(x-5)}{3} + \frac{(x-1)(x-2)(x-4)}{2}$$ ok

$$= x^3 - \frac{29}{3}x^2 + 28x - \frac{52}{3}$$ (optional)

Q7 $\frac{dy}{dx} = f(x,y)$, $y(0) = y_0$, $x_j = jh$, $y(x_j) \approx y_j$.

Explicit Euler $y_{n+1} = y_n + h f(x_n, y_n)$ m2

Implicit Euler $y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$

$n = 0, 1, 2, \ldots$

With $h = 0.1$, $x_0 = 0$, for $f = \sin(x+y-2)$:

$y_1 = y_0 + h f(x_0, y_0) = 3.084$

$y_2 = y_1 + h f(x_1, y_1) = 3.1768 \approx y(0.2)$ m6 A3 + A3

Section B

Q8

i) $L = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 2 & 0 & \\ & & -1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 3 & & & \\ & 4 & & \\ & & 4 & \\ & & & 3 \end{pmatrix}$, $U = \begin{pmatrix} 0 & 1 & & \\ & 0 & 2 & \\ & & 0 & -1 \\ & & & 0 \end{pmatrix}$, m

$$L+D = \begin{pmatrix} 3 & & & \\ 1 & 4 & & \\ & 2 & 4 & \\ & & -1 & 3 \end{pmatrix} = D\begin{pmatrix} 1 & & & \\ \frac{1}{4} & 1 & & \\ 0 & \frac{1}{2} & 1 & \\ 0 & 0 & -\frac{1}{3} & 1 \end{pmatrix}.$$

$$\therefore\ (L+D)^{-1} = \begin{pmatrix} 1 & & & \\ \frac{1}{4} & 1 & & \\ & \frac{1}{2} & 1 & \\ & & -\frac{1}{3} & 1 \end{pmatrix}^{-1} D^{-1},$$

It remains to find the ~~ist~~ inverse from

$$\begin{pmatrix} 1 & & & \\ \frac{1}{4} & 1 & & \\ & \frac{1}{2} & 1 & \\ & & -\frac{1}{3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ \frac{1}{4} & 1 & & \\ & \frac{1}{2} & 1 & \\ & & -\frac{1}{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & \frac{1}{2} & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -\frac{1}{3} & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & & & \\ \frac{1}{4} & 1 & & \\ & \frac{1}{2} & 1 & \\ & & -\frac{1}{3} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & \frac{1}{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & -\frac{1}{2} & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ -\frac{1}{4} & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$ m_2

$$= \begin{pmatrix} 1 & & & \\ -\frac{1}{4} & 1 & & \\ \frac{1}{8} & -\frac{1}{2} & 1 & \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{3} & 1 \end{pmatrix}$$ m

$$\therefore\ (L+D)^{-1} = \begin{bmatrix} \frac{1}{3} & & & \\ -\frac{1}{12} & \frac{1}{4} & & \\ \frac{1}{24} & -\frac{1}{8} & \frac{1}{4} & \\ \frac{1}{72} & -\frac{1}{24} & \frac{1}{12} & \frac{1}{3} \end{bmatrix}.$$

Further from $(L+D)x^{k+1} = -Ux^k + b$ m1

[Note $Ax = (L+D)x + Ux$]

We have $T_{GS} = -(L+D)^{-1}U$

$$= \begin{bmatrix} \frac{1}{3} & & & \\ -1/12 & 1/4 & & \\ 1/24 & -1/8 & \frac{1}{4} & \\ 1/72 & -1/24 & 1/12 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & -1 & & \\ & 0 & -2 & \\ & & 0 & +1 \\ & & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1/3 & 0 & 0 \\ 0 & 1/12 & -1/2 & 0 \\ 0 & -1/24 & 1/4 & 1/4 \\ 0 & -1/72 & 1/12 & 1/12 \end{bmatrix}.$$ m2

ii) The Gerschgorin disks are

$$\begin{cases} |\lambda_1| \le \frac{1}{3} \\ |\lambda_2 - \frac{1}{12}| \le \frac{1}{2} \longrightarrow |\lambda_2| \le \frac{7}{12} = 0.5833 \\ |\lambda_3 - \frac{1}{4}| \le \frac{1}{24} + \frac{1}{8} = 0.1667 \\ |\lambda_4 - \frac{1}{12}| \le \frac{1}{72} + \frac{1}{12} = 0.0972 \end{cases}$$ m6

$= F_4 + A_2$

So $|\lambda(T_{GS})| \le \underline{0.5833} < 1$ and the GS method must converge.

Q9 i) Repeat Q2 for r.e.f. reduction

$$U = \begin{pmatrix} -3 & -6 & 12 \\ & 3 & -6 \\ & & 5 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & & \\ \frac{2}{3} & 1 & \\ -2 & \frac{1}{3} & 1 \end{pmatrix}$$

signs changed in m_{ij}

Q9 i) Obtain L from changing signs of l_{ij}

m2

$$L=\begin{pmatrix}1&&&\\-1/3&1&&\\0&1/3&1&\\-1/3&-2/3&-21&1\end{pmatrix} \text{ and } U=\begin{pmatrix}-3&-6&12&0\\&-3&6&0\\&&-1&-1\\&&&-42\end{pmatrix}$$

for $A=LU$ decomposition. Set $U=DM$.

m3

Then
$$D=\begin{pmatrix}-3&&&\\&-3&&\\&&-1&\\&&&-42\end{pmatrix} \text{ and } M=\begin{pmatrix}1&2&-4&0\\&1&-2&0\\&&1&1\\&&&1\end{pmatrix}.$$

for $A=LDM$.

ii) Solve $Ax=b$ via $LUx=b$ or $Ly=b$, $Ux=y$:

So $y=\begin{pmatrix}75\\42\\-5\\0\end{pmatrix}$ and $x=\begin{pmatrix}3\\-4\\5\\0\end{pmatrix}$

m4 $=A_2+A_2$

iii) $\|A\|_\infty=\max(21,4,3,39)=39$, $\|b\|_1=75+17+9+52=153$.

$\therefore \|A^{-1}\|_\infty \|A\|_\infty=\frac{209}{189}\times 39=\frac{209}{63}\times 13=\frac{2717}{63}$ (≈ 43.127)
$=\kappa_\infty(A)$.

m6 $=A_4+A_3$

Q10 The SIPM is the following

$z=z^{(0)}$

Iterate
$$\begin{cases}(A-\gamma I)y=z\\ \mu=y^{(m)} \text{ if } |y^{(m)}|=\max_j |y^{(j)}|\\ z=y/\mu\end{cases}$$

m4 $=A_1 x$

$\mu \to$ e'value of $(A-\gamma I)^{-1}$ or $\frac{1}{\lambda(A)-\gamma}$

$z \to$ e'vector

With $z = z^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$ and $\gamma = -8$,
the LU factorization is used for solving $(A + 8I)y = z$ so | m1

<u>Step 1</u> From $(A+8I)y = z$
we get

$$\begin{cases} y = (-6.9231, -15.9231, 159.2308)^T \\ \mu = 159.2308, \\ z = y/\mu = (-0.0435, -0.1, 1)^T \end{cases}$$ | A_4

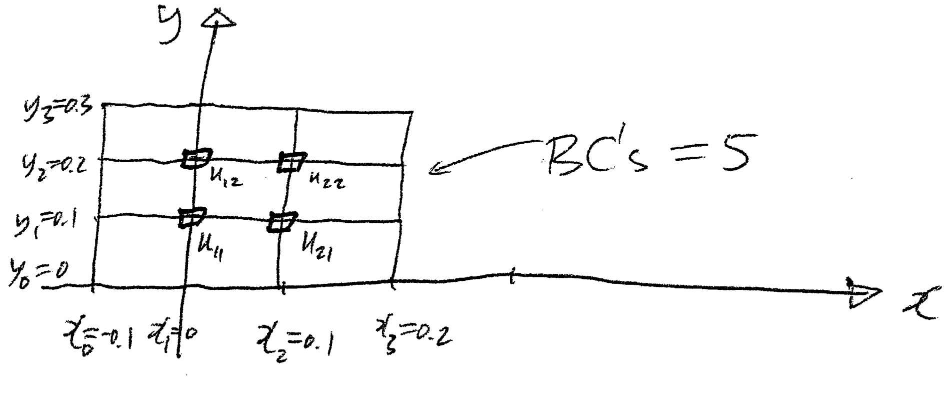
<u>Step 2</u> From $(A+8I)y = z$

$$\begin{cases} y = (-0.7649, -1.7649, 17.5488)^T \\ \mu = 17.5488 \\ z = y/\mu = (-0.0436, -0.1006, 1)^T \end{cases}$$ | A_4

$\therefore \mu = \frac{1}{\lambda(A)+8}$ and $\lambda(A) = -8 + \frac{1}{\mu}$
$= -7.9430$ | m2

and z as eivector ※

<u>Q11</u>



First eqn at $(x_1, y_1) = (0, 0.1)$ (Note $h = \Delta x = \Delta y = 0.1$)

$$(1+y_1)\frac{u_{21}+5-2u_{11}}{h^2} + (1+x_1)\frac{u_{12}+5-2u_{11}}{h^2} = (x_1+y_1+1)^2$$

i.e. $110u_{21} - 420u_{11} + 100u_{12} = 1.21 - 1050 = -1048.79$

as verified.

2nd eqn at $(x_2, y_1) = (0.1, 0.1)$

$$(1+y_1)\frac{u_{11}+5-2u_{21}}{h^2} + (1+x_2)\frac{u_{22}+5-2u_{21}}{h^2} = (x_2+y_1+1)^2$$

i.e. $110u_{11} - 440u_{21} + 110u_{22} = -1098.56$,

Third eqn at $(x_1, y_2) = (0, 0.2)$

$$120u_{22} - 460u_{12} + 100u_{11} = -1098.56,$$

Fourth eqn at $(x_2, y_2) = (0.1, 0.2)$ as

$$120u_{12} - 460u_{22} + 110u_{21} = -1498.31.$$

So the LS is

$$\begin{bmatrix} -420 & 110 & 100 & 0 \\ 110 & -440 & 0 & 110 \\ 100 & 0 & -460 & 120 \\ 0 & 110 & 120 & -460 \end{bmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{pmatrix} = \begin{pmatrix} -1048.79 \\ -1098.56 \\ -1098.56 \\ -1498.31 \end{pmatrix}$$

※

Q12 Let $f = x^0 = 1 \longrightarrow$

$$\int_{-1}^{1} f\,dx = 2 = W_0 + W_1 + W_2$$

$f = x^1 \longrightarrow$

$$\int_{-1}^{1} f\,dx = 0 = W_2 - W_0$$

$f = x^2$

$$\int_{-1}^{1} f\,dx = \frac{2}{3} = W_0 + W_2$$

$A_2 \times 3 = M6$

Solving the 3 equations gives

$W_0 = W_2 = \frac{1}{3}$ and $W_1 = \frac{4}{3}$.

$\therefore$ The rule is $\int_{-1}^{1} f(x)\,dx = \frac{1}{3}\left[f(-1) + 4f(0) + f(1)\right]$.

(Note: Different from Trapezium — This is the Simpson's rule)

$$I_1 = \int_{-1}^{1} \frac{\cos x}{\sqrt{x^2+2}}\,dx = \frac{\sqrt{3}}{9}\cos(-1) + \frac{2\sqrt{2}}{3}\cos 0 + \frac{\sqrt{3}}{9}\cos 1$$

$$= 1.15077$$

$A3 + F. = M5$

$$I_2 = \int_0^3 f(x)\,dx \xrightarrow[x = \frac{3}{2}(y+1)]{\text{Let } y = \frac{2x}{3} - 1} \frac{3}{2}\int_{-1}^{1} \frac{\cos\left(\frac{3(y+1)}{2}\right)}{\sqrt{\frac{9}{4}(y+1)^2 + 2}}\,dy$$

$$= \frac{3}{2} \times \frac{1}{3} \times \left[g(-1) + 4g(0) + g(1)\right]$$

$$= 0.27293$$

$F_2 + A = M4$

Q13 Firstly $\int_0^1 f(x)\,dx = \left(\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^1\right) f(x)\,dx$

$$\underset{\text{Trapezium}}{=} \frac{1}{4}\left[f(0) + 2f(\tfrac{1}{2}) + f(1)\right]$$

m

Secondly

$$\int_0^1 e^{xy-2} u(y)\,dy = \frac{1}{4}\left[e^{-2}u(0) + 2e^{\frac{x}{2}-2}u(\tfrac{1}{2}) + e^{x-2}u(1)\right]$$

m

So letting $u = (u(0),\ u(\tfrac{1}{2}),\ u(1))^T$ and collocating at $0, \frac{1}{2}, 1$ give rise to

$$\begin{bmatrix} 5-\frac{1}{4}e^{-2} & -\frac{1}{2}e^{-2} & -\frac{1}{4}e^{-2} \\ -\frac{1}{4}e^{-2} & 5-\frac{1}{2}e^{\frac{1}{4}-2} & -\frac{1}{4}e^{\frac{1}{2}-2} \\ -\frac{1}{4}e^{-2} & -\frac{1}{2}e^{\frac{1}{2}-2} & 5-\frac{1}{4}e^{1-2} \end{bmatrix} u = \begin{pmatrix} 3 \\ \frac{1}{2}+3 \\ 1+3 \end{pmatrix}$$

mic
$=A_1x$
$+RH$

i.e.

$$\begin{bmatrix} 4.9662 & -0.0677 & -0.0338 \\ -0.0338 & 4.9131 & -0.0558 \\ -0.0338 & -0.1116 & 4.9080 \end{bmatrix} u = \begin{pmatrix} 3 \\ 3.5 \\ 4 \end{pmatrix}$$