SECTION A

1.

Consider the following real-valued function

$$f(x) = \cos(x) - x.$$

(i) Compute $f'(x)$ and explain why $f(x)$ has only one root in $(0, 1)$. [3 marks]

(ii) Use 2 steps of the Secant method to find an approximate solution to $f(x) = 0$, starting from $x^{(0)} = 0, x^{(1)} = 1$.
(Keep at least 4 decimal digits in calculations.) [6 marks]

2.

For the following three simultaneous nonlinear equations in x_1, x_2, x_3

$$\begin{cases} \ln(x_1/2) + x_2 = 0, \\ x_1x_3 + 1 = 0, \\ x_2^2 + 5x_3^2 - 1 = 0. \end{cases}$$

(i) Work out the general Jacobian matrix, J. [2 marks]

(ii) Set up the general formula for using the Newton-Raphson method with some initial guess $\mathbf{x}^{(0)}$. (Do NOT invert any matrices.) [2 marks]

(iii) At the point $\mathbf{x}^{(0)} = [2, 0, -1/2]^T$, compute the Jacobian matrix J and carry out 1 step of the Newton Raphson method. [4 marks]

3.

Consider the linear system $Ax = b$ where

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 4 & 3 & 5 \\ 0 & -3 & 4 \end{pmatrix}, b = \begin{pmatrix} -2 \\ -1 \\ -4 \end{pmatrix}.$$

(i) Solve it by Gaussian elimination. [5 marks]

(ii) Hence find the remaining entries L_{32}, D_{11} and M_{23} in the LDM decomposition of A:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & L_{32} & 1 \end{pmatrix} \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & M_{23} \\ 0 & 0 & 1 \end{pmatrix}.$$

[3 marks]

(iii) Use the LDM decomposition to solve

$$\begin{pmatrix} 2 & 0 & 4 \\ 4 & 3 & 5 \\ 0 & -3 & 4 \end{pmatrix} x = \begin{pmatrix} 4 \\ 11 \\ -2 \end{pmatrix}.$$

[3 marks]

4.

(a) For the following matrix

$$A = \begin{pmatrix} 37 & 1 & 2 \\ 1 & -57 & 5 \\ 2 & 5 & 22 \end{pmatrix}$$

use the Gerschgorin theorem to locate all three eigenvalues and determine if the matrix is SPD (symmetric positive definite). [6 marks]

(b) Let $B = A - \gamma I$. If the shifted inverse power method (for A with the shift $\gamma = 3$) produces the converging sequence of μ_j such that

$$\lim_{j\to\infty} \mu_j = \mu = 3,$$

which eigenvalue $\lambda(B)$ and which corresponding eigenvalue $\lambda(A)$ have been found? [2 marks]

5. [8 marks]

State both the explicit and implicit Euler methods for the following general initial value problem.

$$\frac{dy}{dx} = f(x, y(x)), \quad y(0)) = y_0.$$

Use the explicit Euler method to solve the initial value problem

$$\frac{dy}{dx} = \ln(x + y), \quad y(0) = 2,$$

to obtain y(0.2) with the step length $h = 0.1$. (Keep at least 4 decimal places in calculations.)

6. [11 marks]

Given the linear system $Ax = b$ with

$$A = \begin{pmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \ b = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

write out three equations by the Gauss-Seidel (GS) method to obtain the new iterate $x^{(n+1)}$ from the current iterate $x^{(n)}$. Carry out 2 iterations starting from $x^{(0)} = [1, 0, 0, 1]^T$. (Keep 2 decimal places in calculations.)
Will this method converge? (Explain your answer)

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SECTION B

7. [15 marks]

State the shifted inverse power method to compute an eigenvalue of a matrix A near a given value γ.
Use the shifted inverse power method for 2 steps to estimate both the eigenvalue near $\gamma = 7$ and its corresponding eigenvector for the following matrix:

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & 7 \end{pmatrix}.$$

Start the iteration from $x^{(0)} = [0, 0, 1]^T$ and keep at least 4 decimal places throughout your calculations.
You may use the LU factorization for $(A - 7I)$ i.e.

$$(A - 7I) = \begin{pmatrix} -4 & 0 & 1 \\ 2 & -8 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -1/8 & 1 \end{pmatrix} \begin{pmatrix} -4 & 0 & 1 \\ 0 & -8 & -1/2 \\ 0 & 0 & -1/16 \end{pmatrix}.$$

8.

Consider the solution of the following boundary value problem, by the usual finite difference method with 3×3 boxes, i.e. 4 interior and uniformly distributed mesh points:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2(x+y)^2, \quad p = (x, y) \in \Omega$$

where the domain is the square $\Omega = [0, 0.3] \times [0, 0.3] \in R^2$, with the Dirichlet boundary condition $u = x$ given.

(i) Sketch the computational domain and compute the boundary values.

[5 marks]

(ii) Set up the linear system for the four interior unknowns (*without* having to solve it). Keep at least 4 decimal places throughout your calculations.

[10 marks]

9.

(a) Compute the Lagrange interpolating polynomial $y = P_3(x)$ of degree 3 that passes through these 4 points (x_j, y_j)

$$(0,2),\ (\sqrt{2},3),\ (-1,\sqrt{7}),\ (5,\sqrt{13})$$

[3 marks]

(b) The three point quadrature formula can be written as

$$\int_{-1}^{1} f(x)dx = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2)$$

where $x_0 = -\sqrt{15}/5,\ x_1 = 0,\ x_2 = \sqrt{15}/5.$

Find suitable weights w_j (in exact arithmetic) so that the rule becomes a Gauss type, i.e. the rule is exact for degree $0, 1, 2$ polynomials.

[5 marks]

Adapt the above rule designed for $[-1, 1]$ to approximate (keeping at least 4 decimal digits in calculations)

$$I = \int_0^1 \frac{3x^3 dx}{\sqrt{2 + 10x^4}}.$$

[7 marks]

10.

Consider the linear system $Ax = b$ with

$$A = \begin{pmatrix} -1 & 2 & 0 & 1 \\ 3 & 4 & 10 & -1 \\ 2 & 5 & 2 & 2 \\ 7 & -1 & -1 & 3 \end{pmatrix}, b = \begin{pmatrix} -2 \\ 14 \\ 2 \\ 3 \end{pmatrix}.$$

(i) Use the matrix form of the Gauss elimination method

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -9/10 & 1 & 0 \\ 0 & -13/10 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \end{pmatrix} A$$
$$= \begin{pmatrix} -1 & 2 & 0 & 1 \\ 0 & 10 & 10 & 2 \\ 0 & 0 & -7 & 11/5 \\ 0 & 0 & 0 & 3 \end{pmatrix},$$

to form both the $A = LU$ and the $A = LDM$ decompositions. [5 marks]

(ii) Use the LU decomposition of A to find the solution x. [5 marks]

(iii) Compute $\|A\|_\infty$ and $\|b\|_1$. Using $\|A^{-1}\|_\infty = 8/3$, find the ∞-norm condition number, $\kappa_\infty(A)$. [5 marks]

11.

Given that $Ax = b$,where

$$A = \begin{pmatrix} 5 & -1 & 2 \\ 1 & -3 & -1 \\ 1 & 0 & 7 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 11 \\ 7 \end{pmatrix}.$$

Using exact arithmetic (i.e. fractions):

(i) Write down the 3 equations for the 3 components of the vector $x^{(n+1)}$ for the Jacobi iteration method and carry out 2 iterations starting from $x^{(0)} = \mathbf{0}$. Find the iteration matrix T_J and the vector c_J such that

$$x^{(n+1)} = T_J x^{(n)} + c_J.$$

[7 marks]

(ii) Find $(L + D)^{-1}$, where L and D are the lower diagonal and the diagonal parts of A respectively. Hence compute the iteration matrix T_{GS} of the Gauss-Seidel iteration method such that

$$x^{(n+1)} = T_{GS} x^{(n)} + c_{GS}.$$

[7 marks]

(iii) What necessary and sufficient conditions can you use to check whether each of the Jacobi and Gauss-Seidel Iteration Methods converge or not?

[1 marks]