MATH 266 June 2003

NUMERICAL ANALYSIS, SOLUTION OF LINEAR EQUATIONS

TIME ALLOWED: TWO HOURS AND A HALF

Instructions to candidates

Full marks may be obtained for FIVE complete answers.

All questions will be marked but only the best **five** counted

This examination contributes 70% towards the final mark. The balance comes from coursework which consists of a set of mini projects each of which contains some theory and some computer practical work.

1. a) Two numbers a and b have associated absolute errors of $\pm\epsilon$ and $\pm\eta$ respectively. Write down the absolute and relative errors in $a+b$, $a-b$, $a*b$ and a/b.

b) Given that the maximum possible relative error in the value of a quantity x is ϵ, show that the maximum possible relative error in x^n is $n\epsilon$.

c) Find the largest sized term in the series expansion of the Sine Integral $\mathrm{Si}(x)$ given by

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt = x - \frac{x^3}{3.3!} + \frac{x^5}{5.5!} - \frac{x^7}{7.7!} + \ldots$$

when x has the value $x = 9.0$. If the relative error in this value of x is 10^{-6}, find the absolute error in the largest sized term in the series expansion. What will this contribute to the relative error in the value of $\mathrm{Si}(9.0)$. [The value of $\mathrm{Si}(9.0)$ is 1.6650400758...]

d) For large values of x, $\mathrm{Si}(x)$ may be calculated from the asymptotic series expansion

$$\mathrm{Si}(x) = \frac{\pi}{2} - \frac{1}{x}\left\{1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \frac{6!}{x^6} + \ldots\right\}\cos x - \frac{1}{x}\left\{\frac{1}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \frac{7!}{x^7} + \ldots\right\}\sin x.$$

Show that each of the series in curly brackets diverges for all non-zero values of x. Given that the error in approximating $\mathrm{Si}(x)$ by all the terms in this series up to the smallest is no larger than this smallest term, find the absolute error in using this series to calculate the value of $\mathrm{Si}(4\pi)$.

[20 marks]

2. a) Give a brief derivation of the Newton-Raphson method to find a solution to the equation $f(x) = 0$. Discuss the strengths and weaknesses of the method.

Show that if e_n, the error at the nth stage of the iteration process is small enough and $f'(x) \neq 0$ at the root, the error at the $n+1$ th stage is proportional to e_n^2. Show that if $f'(x) = 0$ at the root, e_{n+1} is proportional to e_n.

b) Describe the Secant Method for finding a solution to the equation $f(x) = 0$. Compare the advantages and disadvantages of this method to other methods of finding a solution to the equation $f(x) = 0$.

[20 marks]

3. Find a lower triangular matrix L with all its diagonal elements equal to 1 and zeros everywhere above the leading diagonal and an upper triangular matrix U with zeros everywhere below the leading diagonal such that the matrix A given below can be written as the product $A = L.U$.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 5 & 2 & 0 & 0 \\ 0 & 2 & 5 & 2 & 0 \\ 0 & 0 & 2 & 5 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}.$$

Show that the matrix U can be written as the product $D.M$, where D is a diagonal matrix whose diagonal elements are 1, 1, 1, 1 and -3 and M is the transpose of the matrix L.

Find the inverses of the matrices L, D and M [You may assume that the inverse of the matrix M is the transpose of the inverse of the matrix L.] Hence or otherwise show that the inverse of the matrix A is

$$A^{-1} = \begin{pmatrix} -1/3 & 2/3 & -4/3 & 8/3 & -16/3 \\ 2/3 & -1/3 & 2/3 & -4/3 & 8/3 \\ -4/3 & 2/3 & -1/3 & 2/3 & -4/3 \\ 8/3 & -4/3 & 2/3 & -1/3 & 2/3 \\ -16/3 & 8/3 & -4/3 & 2/3 & -1/3 \end{pmatrix}.$$

Explain the significance of the condition number of a matrix. Using whichever norm you like, find the condition number for the matrix A.

[20 marks]

4. State Gerschgorin's circle theorem on the eigenvalues of a matrix.

Draw a diagram to show the Gerschgorin circles for the matrix A below. Show that the largest and smallest sized eigenvalues are real and find the largest and smallest values each could have.

$$A = \begin{pmatrix} 76.3 & 13.5 & 1.8 & -11.4 \\ -6.3 & -224.7 & 1.6 & 8.9 \\ 2.2 & -9.8 & 85.7 & -11.2 \\ 4.7 & -8.6 & 1.1 & 8.2 \end{pmatrix}$$

Describe the power method for obtaining the largest eigenvalue of a matrix and explain the theory of how it works. Explain briefly what would happen to the convergence if the largest sized eigenvalue were complex?

Starting with the vector $(0, 1, 0, 0)^T$, perform two iterations of the power method and estimate the value of the largest sized eigenvalue.

Describe the method of inverse iteration for finding eigenvalues of a matrix. Explain how you would use it to evaluate the eigenvalue of smallest size of the matrix A.

[20 marks]

5. Describe the Jacobi method and the Gauss-Seidel method for finding a solution to the set of linear equations $A\mathbf{x} = \mathbf{b}$. What conditions are sufficient to ensure that these methods converge?

Show that both the Jacobi and Gauss-Seidel methods will converge for the matrix A and column vector $\mathbf{b}$, where

$$A = \begin{pmatrix} 200.0 & -5.4 & 6.6 \\ 6.7 & 100.0 & 7.4 \\ 5.5 & 13.0 & -100.0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 410.0 \\ 106.0 \\ 78.0 \end{pmatrix}.$$

Starting with the vector $(2, 1, -1)^T$, perform three iterations using the Gauss-Seidel method.

[20 marks]

6. a) Describe Euler's Method for finding an approximation to the solution of the differential equation $dy/dt = f(t, y)$ with the initial condition $y(a) = \alpha$. Show that the error in the value of $y(b)$, $b > a$ is proportional to $h = (b-a)/N$, where h is the step length and N is the number of steps between a and b, provided that the function $f(t, y)$ is suitably well behaved.

Show that

$$\frac{d^2y}{dt^2} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f.$$

Describe the Second Order Taylor Series method for finding an approximate solution to the differential equation $dy/dt = f(t, y)$. Deduce that the error in $y(b)$ is proportional to h^2, where h is the step length.

b) To use the Finite Difference method to find an approximate solution to the second order equation

$$\frac{d^2y}{dx^2} = p(x)\frac{dy}{dx} + q(x)y + r(x),$$

with boundary conditions $y(x) = \alpha$ and $y(b) = \beta$, where $a < b$, we divide the interval (a, b) up into $N + 1$ equal strips of width $h = (b - a)/(N + 1)$. We denote $x_i = a + ih$, where $0 \leq i \leq N + 1$ and $y_i = y(x_i)$. Find approximate expressions for d^2y/dx^2 and dy/dx in terms of y_{i+1}, y_i, y_{i-1} and h.

Write down the form of the matrices A and $\mathbf{b}$ when the Finite Difference method approximation to the boundary value problem

$$\frac{d^2y}{dx^2} = (1 - x^2)y - \frac{1}{1 + x^2} \qquad y(-1) = 0, \quad y(1) = 1$$

with 201 points is expressed as the linear equation

$$A.\mathbf{y} = \mathbf{b}.$$

[20 marks]

7. a) The Maclaurin series for a function $f(x)$ to second order with remainder term can be written:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(\xi(x)),$$

where $0 < \xi < x$.

Integrate the above expression from $-h$ to h to show that

$$\int_{-h}^{h} f(x)dx = 2hf(0) + \frac{h^3}{3} f''(\eta) \qquad -h < \eta < h.$$

This rule is known as the mid-point rule. How would you adapt this rule to evaluate the integral $\int_a^b f(x)dx$ using n steps?

Show that if $f(x) = e^{-x^2}$, for $0 < x < 1$, $|f''(x)| < 2$. Use the error estimate above to determine the number of steps necessary to ensure that the error in the evaluation of $\int_0^1 e^{-x^2} dx$ by this method is no more than 1/12000.

Could you use this method to evaluate the integral

$$\int_0^1 \frac{e^{-x}}{\sqrt{x}} dx?$$

Transform the above integral into a more suitable form in order that the above method can be used to evaluate it.

b) Determine the values of the constants α, β, x_1 and x_2 such that the quadrature rule for the integral

$$\int_{-1}^{1} f(x)\, dx = \alpha f(x_1) + \beta f(x_2)$$

is exact for $f(x) = 1$, $f(x) = x$, $f(x) = x^2$ and $f(x) = x^3$.

[20 marks]