SECTION A

1. Consider the following real-valued function

$$f(x) = \cos(x) - x.$$

(i) Compute $f'(x)$ and verify that $f'(x) > 0$ in $(0, 1)$. How many roots can $f(x)$ have in $(0, 1)$?

[3 marks]

(ii) Use 2 steps of the Newton-Raphson method to find an approximate solution to $f(x) = 0$, starting from $x^{(0)} = 0$.
(Keep at least 4 decimal digits in calculations.)

[4 marks]

(iii) Starting from $x^{(0)} = 0$ and $x^{(1)} = 1$, use 2 steps of the Secant method to find an approximate solution to $f(x) = 0$. (Keep at least 4 decimal digits in calculations.) What advantage does the Secant method have over the Newton-Raphson method?

[5 marks]

2. Sketch the following two simultaneous nonlinear equations in x_1, x_2,

$$\begin{cases} (x_1 - 1)^2 + x_2^2 = 4, \\ (x_1 - 3)^2 + \frac{1}{9}x_2^2 = 1. \end{cases}$$

Taking the initial guess $x^{(0)} = (2, 2)^T$, use 1 step of the Newton-Raphson method to find an approximate solution. (Keep at least 4 decimal digits in calculations.)

[10 marks]

3. Consider the linear system $Ax = b$ with

$$A = \begin{pmatrix} 4 & 1 & 8 \\ 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}.$$

(i) Solve it by Gaussian elimination using exact arithmetic (i.e. fractions).

[4 marks]

(ii) Hence or otherwise find the LDM decomposition of A.

[6 marks]

4. Consider the matrix A

$$A = \begin{pmatrix} 12 & 1 & 2 \\ 4 & -2 & 1 \\ 0 & 2 & 1 \end{pmatrix}.$$

(i) Use the Gerschgorin theorem to locate the eigenvalues.

[4 marks]

(ii) Use two steps of the power method to estimate the dominant eigenvalue. Take the initial guess $z = [1, 0, 0]^T$. (Keep at least 2 decimal places in calculations.)

[6 marks]

5. Compute the Lagrange interpolating polynomial $y = P_3(x)$ of degree 3 that passes through these 4 points (x_j, y_j)

$$(0, 2),\ (\sqrt{2}, 3),\ (-1, \sqrt{7}),\ (5, \sqrt{13}).$$

What is the minimum degree polynomial that can pass through 6 arbitrarily chosen points?

[5 marks]

6. State both the explicit and implicit Euler methods for the following general initial value problem.

$$\frac{dy}{dx} = f(x, y(x)), \quad y(0) = y_0.$$

Use the explicit Euler method to solve the initial value problem

$$\frac{dy}{dx} = (x+y)^2 + 3, \quad y(0) = 2,$$

to obtain y(0.2) with the step length $h = 0.1$ (keep at least 4 decimal places in calculations). Compare your result with the exact solution:

$$y(x) = -x + 2\tan(2x + \pi/4).$$

Suggest how you could improve the accuracy of the numerical estimate.

[8 marks]

SECTION B

7. Use the shifted inverse power method for 2 steps to estimate both the eigenvalue near $\gamma = 3$ and its corresponding eigenvector for the following matrix

$$A = \begin{pmatrix} -2 & 2 & -1 \\ 0 & 7 & 0 \\ -1 & 3 & 3 \end{pmatrix}.$$

Start the iteration from $z = [0, 0, 1]^T$ and keep at least 4 decimal places in your calculations. You may use the factorization

$$(A - 3I) = LU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.2 & 0.65 & 1 \end{pmatrix} \begin{pmatrix} -5 & 2 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}.$$

[15 marks]

8. Consider the solution of the following boundary value problem, by the usual finite difference method with 3×3 boxes, i.e. 4 interior and uniformly distributed mesh points:

$$\frac{\partial^2 u}{\partial x^2} + (1 + x^2)\frac{\partial^2 u}{\partial y^2} = \cos(10y), \quad p = (x, y) \in \Omega$$

where the domain is the rectangle $\Omega = [0, 0.3] \times [0, 0.3] \in R^2$, with the Dirichlet boundary condition $u = 2x$ given.

(i) Sketch the computational domain and compute the boundary values.

[5 marks]

(ii) Set up the linear system for the four interior unknowns (*without* having to solve it). Keep at least 4 decimal places throughout your calculations.

[10 marks]

9. Consider the linear system $Ax = b$ where

$$A = \begin{pmatrix} -3 & 1 & 0 \\ 2 & 10 & 1 \\ 0 & 2 & 7 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}.$$

Using exact arithmetic (i.e. fractions):

(i) Write out three equations for the 3 components of the vector $x^{(n+1)}$ for the Jacobi Iteration Method and carry out 2 iterations starting from $x^{(0)} = (0, 0, 0)^T$. Find the iteration matrix T_J and the vector c_J such that

$$x^{(n+1)} = T_J x^{(n)} + c_J.$$

[7 marks]

(ii) Find $(L + D)^{-1}$, where L and D are the lower diagonal and the diagonal parts of A respectively. Hence compute the iteration matrix T_{GS} of the Gauss-Seidel iteration method.

[7 marks]

(iii) What necessary and sufficient conditions can you use to check whether each of the Jacobi and Gauss-Seidel iteration methods converge or not?

[1 marks]

10. The three point quadrature formula can be written as

$$\int_{-1}^{1} f(x)dx = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2),$$

where $x_0 = -1,\ x_1 = 0,\ x_2 = 1$.

Find suitable weights w_j (in exact arithmetic) so that the rule becomes a Gauss type, i.e. the rule is exact for degree $0, 1, 2$ polynomials.

[5 marks]

Use the rule you have obtained to approximate the integral

$$I = \int_{-1}^{1} \frac{e^x dx}{\sqrt{3 + 2x^3}}.$$

(Keep at least 4 decimal digits in calculations.)

[3 marks]

Modify the rule you obtained to approximate the integral

$$I = \int_{0}^{2} \frac{e^x dx}{\sqrt{3 + 2x^3}}.$$

(Keep at least 4 decimal digits in calculations.)

[7 marks]

11. Consider the matrix A:

$$A = \begin{pmatrix} 2 & 5 & -1 & 0 \\ 1 & 2 & 2 & 0 \\ 0 & 0 & 4 & 1 \\ 3 & 6 & 0 & -1 \end{pmatrix}.$$

Using exact arithmetic (i.e. fractions), compute the $PA = LU$ decomposition of matrix A with partial pivoting.

[15 marks]