THE UNIVERSITY
of LIVERPOOL

Math 266

Numerical Analysis, Solution of Linear Systems

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Year 2 Sept 2005 Paper

Full marks will be awarded for complete answers to EIGHT questions, of which

- FIVE answers must be from **Section A** and
- THREE from **Section B**
  (only the best 3 answers from Section B will be taken into account).

SECTION A
(ATTEMPT ALL QUESTIONS)

1.

The nonlinear equation

$$7\ln(2-x) - 7\ln(2+x) + 18 = 0$$

has a solution in either $(0.5, 1.5)$ or $(1.5, 1.9)$. Explain which interval contains the solution and further use 1 step of the Newton-Raphson method to estimate it, with $x^{(0)} = 1.5$. (Keep at least 4 significant digits in your calculations.) [**10 marks**]

2.

For the following simultaneous nonlinear equations for x, y, z

$$\begin{cases} 3x^2 + 7y^2 + z^2 = 100 \\ 6x^2 - 3y^2 + z^2 = 120 \\ \qquad\quad x + y - z = 1, \end{cases}$$

i) Work out the Jacobian matrix J at a general point $\mathbf{x} = [x, y, z]^T$. [**3 marks**]

ii) Verify that the Jacobian matrix at $\mathbf{x} = \mathbf{x}^{(0)} = [4, 2, 5]^T$ is

$$J^{(0)} = \begin{bmatrix} 24 & 28 & 10 \\ 48 & -12 & 10 \\ 1 & 1 & -1 \end{bmatrix}.$$

[**3 marks**]

iii) Given that

$$(J^{(0)})^{-1} = \begin{bmatrix} 1/1136 & 19/1136 & 25/142 \\ 29/1136 & (-17)/1136 & 15/142 \\ 15/568 & 1/568 & (-51)/71 \end{bmatrix},$$

carry out 1 step of the Newton-Raphson method at $\mathbf{x} = \mathbf{x}^{(0)}$. (Keep at least 4 significant digits in your calculations.) [**4 marks**]

3.

For the following matrix

$$A = \begin{bmatrix} 99 & 1 & 1 \\ 1 & -88 & -54 \\ 1 & -54 & 87 \end{bmatrix},$$

use the Gerschgorin theorem to locate all three eigenvalues and determine if the matrix is SPD (symmetric positive definite).

Let $B = A - \gamma I$. If the shifted inverse power method (for A with the shift $\gamma = 4$) produces the converging sequence of μ_j such that

$$\lim_{j\to\infty} \mu_j = \mu = -2,$$

which eigenvalue $\lambda(B)$ and which corresponding eigenvalue $\lambda(A)$ have been found?

[**10 marks**]

4.

i) For $L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 5 & 0 \\ -2 & 1 & 9 \end{pmatrix}$, compute L^{-1} using the factorisation method. Find the condition number of L (in the ∞-norm) and the spectral radius $\rho(L)$?

[**10 marks**]

ii) Compute $\|A\|_1$ and $\|A\|_F$ for $A = \begin{pmatrix} -1 & 0 \\ 1 & \sqrt{2} \end{pmatrix}$. [**5 marks**]

5.

Consider the following linear system

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 \\ -3 \\ 3 \end{pmatrix}.$$

i) Solve it by Gaussian elimination. [**5 marks**]

ii) Hence or otherwise find the LDM decomposition of A. [**5 marks**]

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SECTION B

(CHOOSE ANY THREE QUESTIONS)

6.

Given that $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 7 & 2 & 4 \\ 1 & 5 & 3 \\ 2 & 3 & 6 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 21 \\ 10 \\ 6 \end{pmatrix},$$

i) write down the three equations for the three components of the vector $\mathbf{x}^{(n+1)}$ for the Jacobi iteration method and carry out 2 iterations starting from $\mathbf{x}^{(0)} = \mathbf{0}$. Find the iteration matrix T_J and the vector $\mathbf{c}_J$ such that

$$\mathbf{x}^{(n+1)} = T_J \mathbf{x}^{(n)} + \mathbf{c}_J.$$

Determine whether or not the iteration method converges. [**7 marks**]

ii) Find $(L + D)^{-1}$, where L and D are the lower triangular and the diagonal parts of A respectively. Hence compute the iteration matrix T_{GS} of the Gauss-Seidel iteration method and determine whether or not the iteration method converges. [**8 marks**]

7.

For the following matrix A

$$\begin{bmatrix} 8 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -14 & 8 \end{bmatrix},$$

use the shifted inverse power method for 2 steps to estimate both the eigenvalue near $\gamma = 0.5$ and its corresponding eigenvector. Start from $\mathbf{x}^{(0)} = [-1 \;\; 2 \;\; 1]^T$ and keep at least 4 significant digits in your calculations. [**15 marks**]

Hint. You may use the result

$$(A - 0.5I)^{-1} = \begin{bmatrix} 0.1357 & -0.5068 & -0.0181 \\ 0 & 2.0000 & 0 \\ -0.0181 & 3.8009 & 0.1357 \end{bmatrix}.$$

8.

Consider the following boundary value problem

$$u - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = x^2 + y^2, \qquad (x, y) \in \Omega$$

where the domain is the square $\Omega = [0, 0.3] \times [0, 0.3] \in R^2$, with the Dirichlet boundary condition $u|_{\partial\Omega} = xy$, to be solved by the finite difference (FD) method with 3×3 boxes i.e. 4 interior and uniformly distributed mesh points. Set up the linear system for the 4 interior unknowns (there is no need to *solve* the system). [**15 marks**]

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9.

A general quadrature rule may be denoted by

$$\int_a^b f(x)dx = \sum_{j=0}^{N} w_j f(x_j).$$

i) Write down the middle-point rule (i.e. $N = 0,\ x_0 = 0$) for evaluating

$$\int_{-1}^{1} f(x)dx.$$

By mapping the interval $[0, 1]$ to $[-1, 1]$, use the middle-point rule to evaluate the following integral

$$I = \int_0^1 \left[2008x + \frac{x^2}{\sqrt{10x^3+3}} \right] dx.$$

(Keep at least 4 significant digits in your calculations.)

Verify that the exact value for the definite integral is

$$I = 1004 + \frac{\sqrt{13}-\sqrt{3}}{15}.$$

Compute the absolute error of the approximation to this exact value.

[**8 marks**]

ii) The three-point quadrature rule can be written as

$$\int_{-1}^{1} f(x)dx = w_0\, f(x_0) + w_1\, f(x_1) + w_2 f(x_2),$$

where $x_0 = -1, \quad x_1 = 0, \quad x_2 = 1.$

Verify that for the rule to become a Gauss type, the following must hold

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2/3 \end{pmatrix}.$$

Further find the weights w_0, w_1, w_2. [**7 marks**]

10.

i) Explain how a quadrature rule is usually used to develop a numerical method for solving an initial value problem. [**3 marks**]

ii) State the implicit Euler method for solving

$$\frac{dy}{dx} = f(x, y), \quad x \geq x_0, \qquad\qquad y(x_0) = y_0.$$

[**4 marks**]

iii) Combine the nonlinear Newton-Raphson method with the implicit Euler method to find a solution of the initial value problem

$$\frac{dy}{dx} = ye^{-5x} - y^2, \qquad\qquad y(0) = 1$$

at $x = 0.2$ with the step length $h = 0.1$.
(Use no more than 2 iterations in each Newton-Raphson step.) [**8 marks**]
