THE UNIVERSITY
of LIVERPOOL

SECTION A

1. Consider the following real-valued function

$$f(x) = \ln\left(\frac{1-x}{1+x}\right) + 1.$$

(i) Compute $f'(x)$ and explain why $f(x)$ has exactly one root in $(0,1)$.

[3 marks]

(ii) Use 2 steps of the Newton-Raphson method to find an approximate solution to $f(x) = 0$, starting from $x^{(0)} = 0$. (Keep at least 4 decimal digits in calculations.)

[4 marks]

(iii) Compute the error in the first and second steps, given the exact solution is

$$x = \frac{e-1}{e+1}.$$

Estimate the error in the third step, given that the Newton-Raphson method exhibits quadratic convergence.

[4 marks]

2. Starting from $x^{(0)} = 0$ and $x^{(1)} = 0.5$, solve the nonlinear equation

$$f(x) = \ln\left(\frac{1-x}{1+x}\right) + 1 = 0$$

using 2 steps of the Secant method. (Keep at least 4 decimal digits in calculations.) What is the minimum number of times the function $f(x)$ has to be evaluated for 2 steps of the Secant method?

[5 marks]

3. For the following three simultaneous nonlinear equations in x_1, x_2, x_3

$$\begin{cases} 4x_1 - x_3^2 = 0, \\ x_2^2 + 2x_1^2 - 3 = 0, \\ x_2 x_3 - 6\cos(x_2 + x_3) = 0. \end{cases}$$

(i) Work out the general Jacobian matrix, J.

[2 marks]

(ii) Set up the general formula for using the Newton-Raphson method with some initial guess $x^{(0)}$. (Do NOT invert any matrices.)

[2 marks]

(iii) Carry out 1 step of the Newton Raphson method starting at the point $x^{(0)} = (1, -1, 2)^T$. (Keep at least 4 decimal digits in calculations.)

[6 marks]

4. Consider the linear system $Ax = b$ where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 5 & 4 \\ 2 & 10 & 11 \end{pmatrix}, b = \begin{pmatrix} 4 \\ 12 \\ 15 \end{pmatrix}.$$

(i) Solve it by Gaussian elimination.

[4 marks]

(ii) Hence find the remaining entries L_{23}, D_{33} and M_{12} in the LDM decomposition of A:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ L_{23} & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & D_{33} \end{pmatrix} \begin{pmatrix} 1 & M_{12} & 0 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{pmatrix}.$$

[3 marks]

(iii) Use the LDM decomposition to solve

$$\begin{pmatrix} 2 & 1 & 0 \\ 4 & 5 & 4 \\ 2 & 10 & 11 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

[3 marks]

5. Consider the linear system $Ax = b$ with

$$A = \begin{pmatrix} 10 & 1 & -2 & 0 \\ 0 & 10 & -1 & 3 \\ 0 & -2 & 8 & -1 \\ 0 & 3 & -1 & 5 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 25 \\ 1 \\ 0 \end{pmatrix}.$$

Write out four equations by the Gauss-Seidel (GS) method to obtain the new iterate $x^{(n+1)}$ from the current iterate $x^{(n)}$. Carry out 2 iterations starting from $x^{(0)} = (0, 0, 0, 0)^T$. (Keep 2 decimal places in calculations.)

[7 marks]

6.

(i) Using the composite Trapezium rule (with 2 equal subintervals) approximate

$$\int_0^1 e^{xy-1}u(y)dy.$$

[4 marks]

(ii) Using this approximation for the integral, set up the linear system to find the numerical solution of the following equation

$$2u(x) - \int_0^1 e^{xy-1}u(y)dy = x - 1, \quad x \in [0, 1].$$

Do not solve the system.

[8 marks]

SECTION B

7. Consider the matrix A:

$$A = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 2 & 0 & -1 \\ 4 & 8 & -1 & 0 \\ 0 & 2 & 1 & -3 \end{pmatrix}.$$

Using exact arithmetic (i.e. fractions), compute the $PA = LU$ decomposition of matrix A with partial pivoting.

[15 marks]

8. Consider the matrix A:

$$A = \begin{pmatrix} -5 & 2 & -1 \\ 0 & -1 & 0 \\ -1 & 3 & 7 \end{pmatrix}.$$

(i) Given the result

$$(A + 5I)^{-1} = \begin{pmatrix} -12 & 6.75 & -1 \\ 0 & 0.25 & 0 \\ -1 & 0.5 & 0 \end{pmatrix},$$

use the shifted inverse power method for 2 steps to compute the eigenvalue of A near $\gamma = -5$ and its corresponding eigenvector. Start from $z = [1, 0, 0]^T$ and keep at least 4 decimal digits in calculations.

[12 marks]

(ii) Use the Gerschgorin theorem to locate the other two eigenvalues.

[3 marks]

9.

(i) State both the explicit and implicit Euler methods for the following general initial value problem:

$$\frac{dx}{dt} = f(t, x(t)), \quad x(0) = x_0.$$

[2 marks]

(ii) Use the explicit Euler method to solve the initial value problem

$$\frac{dx}{dt} = 3x(1 - 0.1x), \quad x(0) = 1,$$

to obtain $x(0.2)$ with the step length $h = 0.1$. (Keep at least 4 decimal places in calculations.) Compare your result with the exact solution:

$$x(t) = \frac{10e^{3t}}{10 + (e^{3t} - 1)}.$$

Suggest how you could improve the accuracy of the numerical estimate.

[6 marks]

(iii) Use the implicit Euler method to obtain $x(0.1)$ with the step length $h = 0.1$ for the same initial value problem.

[5 marks]

(iv) Comment on the advantages and disadvantages of the implicit and explicit Euler schemes.

[2 marks]

10. Consider the solution of the following boundary value problem, by the usual finite difference method with 3×3 boxes, i.e. 4 interior and uniformly distributed mesh points:

$$(1+2y)\frac{\partial^2 u}{\partial x^2} + (1+x)\frac{\partial^2 u}{\partial y^2} = (x+y+1)^2, \quad p = (x,y) \in \Omega$$

where the domain is the square $\Omega = [0, 0.3] \times [-0.1, 0.2] \in R^2$, with the Dirichlet boundary condition $u = 2y$ given.

(i) Sketch the computational domain and compute the boundary values.

[5 marks]

(ii) Set up the linear system for the four interior unknowns (*without* having to solve it). Keep at least 4 decimal places throughout your calculations.

[10 marks]

11. Given that $Ax = b$, where

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & -5 & 1 \\ 0 & 1 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 5 \\ 12 \end{pmatrix}.$$

Using exact arithmetic (i.e. fractions):

(i) Write down the 3 equations for the 3 components of the vector $x^{(n+1)}$ for the Jacobi iteration method and carry out 2 iterations starting from $x^{(0)} = (0,0,0)^T$. Find the iteration matrix T_J and the vector c_J such that

$$x^{(n+1)} = T_J x^{(n)} + c_J.$$

[7 marks]

(ii) Find $(L+D)^{-1}$, where L and D are the lower diagonal and the diagonal parts of A respectively. Hence compute the iteration matrix T_{GS} of the Gauss-Seidel iteration method. [7 marks]

(iii) What necessary and sufficient conditions can you use to check whether each of the Jacobi and Gauss-Seidel iteration methods converge or not?

[1 marks]