THE UNIVERSITY
of LIVERPOOL

Math 266

May 2005 Exam

Numerical Analysis: Solution of Linear Systems

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Year 2 Paper

Full marks will be awarded for complete answers to TEN questions, of which

- SEVEN answers must be from **Section A** and
- THREE from **Section B**
  (only the best 3 answers from Section B will be taken into account).

SECTION A

(ATTEMPT ALL QUESTIONS: 55%)

1.

Explain why the following real-valued function

$$f(x) = 5 - 2x + \ln\left(\frac{2+x}{3+x}\right)$$

has a root in the interval $(2, 2.5)$. [4 marks]

Verify that $f'(x) = -\dfrac{2(x+5/2)^2 - 3/2}{(x+2)(x+3)}$ and further use 2 steps of the Newton-Raphson method to find an approximate solution to $f(x) = 0$, starting from $x^{(0)} = 2.5$. (Keep at least 5 decimal places throughout your calculations.) [8 marks]

2.

Use elementary row operations to reduce the following matrix

$$A = \begin{pmatrix} -3 & -6 & 12 \\ -2 & -1 & 2 \\ 6 & 13 & -21 \end{pmatrix}$$

to an upper triangular form. (No calculators required.) [5 marks]

3.

The following simultaneous nonlinear equations

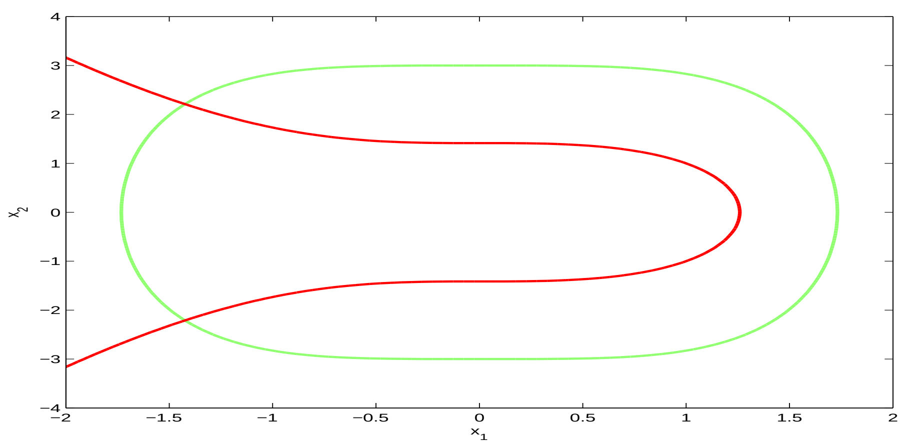
$$\begin{cases} x_1^3 + x_2^2 - 2 = 0, \\ x_1^4 + x_2^2 - 9 = 0, \end{cases}$$

are plotted in Fig.1. Estimate the particular solution which is closest to the point $(-2, 3)$. [2 marks]

Taking the initial guess $\mathbf{x}^{(0)} = (-1, 2)^T$, use 1 step of the Newton-Raphson method to find the solution. [10 marks]

(Keep at least 4 decimal places throughout your calculations.)

Figure 1. Illustration of two curves in 2005 paper



4.

Given the linear system $A\mathbf{x} = \mathbf{b}$ with [8 marks]

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 20 \\ 19 \\ 29 \\ -5 \end{pmatrix},$$

write out the three equations, by the Gauss-Seidel (GS) method, to obtain the new iterate $\mathbf{x}^{(n+1)}$ from the current iterate $\mathbf{x}^{(n)}$. Carry out 2 iterations starting from $\mathbf{x}^{(0)} = [9\ 0\ 9\ 0]^T$.

(Keep at least 4 decimal places throughout your calculations.)

5.

To be able to use the shifted inverse power method, suggest a suitable shift for finding each of the 3 eigenvalues of the following matrix

$$A = \begin{bmatrix} 15 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & -8 \end{bmatrix}.$$

Explain why the eigenvalues must be real.

[5 marks]

6.

Find the Lagrange interpolating polynomial $y = P_3(x)$ of degree 3, which passes through the following 4 points (x_j, y_j):

$$(1,2), \quad (2,8), \quad (4,4), \quad (5,6).$$

[5 marks]

7.

State both the explicit and implicit Euler methods for the following general initial value problem

$$\frac{dy}{dx} = f(x, y(x)), \qquad y(0) = y_0.$$

[2 marks]

Use the explicit Euler method to solve the initial value problem

$$\frac{dy}{dx} = \sin(x + y - 2), \qquad y(0) = 3,$$

to obtain $y(0.2)$ with the step length $h = 0.1$. [6 marks]

SECTION B

(CHOOSE ANY THREE QUESTIONS: 45%)

8.

For the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 20 \\ 19 \\ 29 \\ -5 \end{pmatrix},$$

i) write down L, D and U, the lower triangular, the diagonal and the upper triangular parts of A respectively. Find $(L+D)^{-1}$ and hence obtain the iteration matrix T_{GS} such that

$$\mathbf{x}^{n+1} = T_{GS}\mathbf{x}^n + \mathbf{c}_{GS},$$

where the vector $\mathbf{c}_{GS} = [20/3 \;\; 37/12 \;\; 137/24 \;\; 17/72]^T$;
(No calculators required.) [9 marks]

ii) use the Gerschgorin theorem to determine whether or not the GS method converges, assuming all the eigenvalues of T_{GS} are real. [6 marks]

9.

Consider the linear system $A\mathbf{x} = \mathbf{b}$ with

$$\begin{pmatrix} -3 & -6 & 12 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & -1 \\ 1 & 4 & 13 & -21 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 75 \\ 17 \\ 9 \\ 52 \end{pmatrix}.$$

i) Use the matrix form of the Gauss elimination method

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 21 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1/3 & 1 & 0 \\ 0 & 2/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 1 \end{pmatrix} A$$

$$= \begin{pmatrix} -3 & -6 & 12 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -42 \end{pmatrix}$$

to form both the LU and the LDM decompositions:
$A = LU$ and $A = LDM$. [5 marks]

ii) Use the LU decomposition of A to find the solution $\mathbf{x}$. [4 marks]

iii) Compute $\|A\|_\infty$ and $\|\mathbf{b}\|_1$. Now using $\|A^{-1}\|_\infty = 209/189$, find the ∞-norm condition number $\kappa_\infty(A)$. [6 marks]

10.

State the shifted inverse power method to compute $\lambda(A)$. [4 marks]

Use the shifted inverse power method for 2 steps to estimate both the eigenvalue near $\gamma = -8$ and its corresponding eigenvector for the following matrix

$$A = \begin{bmatrix} 15 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & -8 \end{bmatrix}.$$

Start the iteration from $\mathbf{x}^{(0)} = [0\ \ 0\ \ 9]^T$ and keep at least 2 decimal places throughout your calculations. [11 marks]

You may use the LU factorisation for $(A + 8I)$ i.e.

$$\begin{bmatrix} 23 & 0 & 1 \\ 0 & 10 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/23 & -1/10 & 1 \end{bmatrix} \begin{bmatrix} 23 & 0 & 1 \\ 0 & 10 & 1 \\ 0 & 0 & 13/230 \end{bmatrix}.$$

11.

Consider the following boundary value problem

$$(1+y)\frac{\partial^2 u}{\partial x^2} + (1+x)\frac{\partial^2 u}{\partial y^2} = (x+y+1)^2, \qquad (x,y) \in \Omega$$

where the domain is the square $\Omega = [-0.1, 0.2] \times [0, 0.3] \in R^2$, with the Dirichlet boundary condition $u = 5$ on all boundary points, to be solved by the finite difference (FD) method with 3×3 boxes i.e. 4 interior and uniformly distributed mesh points.

Verify that the FD equation at the point $(x_1, y_1) = (0, 0.1)$ is

$$110u_{21} - 420u_{11} + 100u_{12} = -1048.79.$$

Find the remaining three FD equations and the final linear system (there is no need to *solve* the system).

Keep at least 2 decimal places throughout your calculations.

[15 marks]

12.

Design a three-point quadrature rule of the Gauss type [6 marks]

$$\int_{-1}^{1} f(x)dx = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2),$$

by choosing suitable weights w_0, w_1, w_2, where $x_0 = -1, \quad x_1 = 0, \quad x_2 = 1$.
(*Hint. The rule should be exact for polynomials of degree* $0, 1, 2$.)

Use the rule you obtained to approximate the integral [5 marks]

$$I_1 = \int_{-1}^{1} \frac{\cos x}{\sqrt{x^2+2}} dx.$$

Modify the rule you obtained to approximate the integral

$$I_2 = \int_{0}^{3} \frac{\cos x}{\sqrt{x^2+2}} dx.$$

(Keep at least 4 decimal places in all calculations.) [4 marks]

13.

Using the composite Trapezium rule (with 2 subintervals) in a collocation method, set up the linear system to find the numerical solution of the following integral equation

$$5u(x) - \int_{0}^{1} e^{xy-2} u(y)dy = x + 3, \qquad x \in [0, 1].$$

(No need to solve the system and no calculators required.)

[15 marks]