THE UNIVERSITY
of LIVERPOOL

Math 266

May 2004 Exam

Numerical Analysis: Solution of Linear Systems

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Year 2 Paper

Full marks will be awarded for complete answers to NINE questions, of which

- SIX answers must be from **Section A** and
- THREE from **Section B**
(only the best 3 answers from Section B will be taken into account).

SECTION A

(ATTEMPT ALL QUESTIONS)

1. **[10 marks]**

For the following real-valued function

$$f(x) = \ln\left(\frac{1-x}{1+x}\right) + 2,$$

i) compute $f'(x)$ and verify that $f'(x) < 0$ in $(-1, 1)$. How many roots can $f(x)$ have in $(-1, 1)$?

ii) use 2 steps of the Newton-Raphson method to find an approximate solution to $f(x) = 0$, starting from $x^{(0)} = 0.5$.
(Keep at least 4 decimal digits in calculations.)

2. **[8 marks]**

Starting from $x^{(0)} = 0.5$ and $x^{(1)} = 0.6$, solve the nonlinear equation

$$\ln\left(\frac{1-x}{1+x}\right) + 2 = 0$$

using 2 steps of the Secant method. (Keep at least 4 decimal digits in calculations.)

3. **[8 marks]**

For the following three simultaneous nonlinear equations in x_1, x_2, x_3

$$\begin{cases} \sin(x_1 + x_2) - x_1 x_2 = 0, \\ x_1^2 + x_2^2 - 1 = 0, \\ 5x_1^2 + 21x_2^2 - 9x_3 = 0, \end{cases}$$

i) work out the general Jacobian matrix J;

ii) set up the general formula for using the Newton-Raphson method with some initial guess $\mathbf{x}^{(0)}$; (Do not invert any matrices.)

iii) at the point $\mathbf{x}^{(0)} = \begin{bmatrix} 0 & -1 & 2 \end{bmatrix}^T$, compute the Jacobian matrix J and carry out one step of the Newton-Raphson method. (Keep at least 4 decimal digits in calculations.)

4. **[13 marks]**

For the following linear system

$$\begin{pmatrix} 2 & 4 & 2 \\ 1 & 0 & 3 \\ 3 & 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -4 \\ 4 \\ 3 \end{pmatrix},$$

i) solve it by Gaussian elimination. [4 marks]

ii) explain how the multipliers and coefficients would give rise to the LU decomposition. [1 mark]

iii) using i) and ii), find the remaining entries L_{32}, D_{22}, M_{13} in the $A = LDM$ decomposition: [6 marks]

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & L_{32} & 1 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & -6 \end{pmatrix} \text{ and } M = \begin{pmatrix} 1 & 2 & M_{13} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

iv) In the first stage of a Gaussian elimination method with partial pivoting for the above system, what permutation P_1 is appropriate? [2 marks]

5. **[11 marks]**

(a) Compute $\|A\|_\infty$ and $\|b\|_2$ for $A = \begin{pmatrix} -1 & 5 \\ 1 & -1 \end{pmatrix}$, $b = [3 \;\; 0 \;\; 4]^T$.

(b) Compute M^{-1} for $M = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ and verify that the 1-norm condition number of M is $\kappa_1(M) = \mathrm{cond}_1(M) = 24$.

6. **[5 marks]**

For the following matrix A

$$\begin{bmatrix} 9 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -14 & 8 \end{bmatrix},$$

use the Gerschgorin theorem to decide which of the following three values

$$-50, \quad 1.2, \quad 50,$$

is closest to an eigenvalue of A.

SECTION B

(CHOOSE ANY THREE QUESTIONS)

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7. **[15 marks]**

Given that $A\mathbf{x} = \mathbf{b}$,

$$A = \begin{pmatrix} 9 & 2 & 0 \\ 2 & 8 & -3 \\ 0 & -3 & 7 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 13 \\ 15 \\ 1 \end{pmatrix},$$

i) write out the three equations for the three components of the vector $\mathbf{x}^{n+1}$ for the Jacobi Iteration Method and carry out 2 iterations starting from $\mathbf{x}^0 = \mathbf{0}$. Find the iteration matrix T_J and vector $\mathbf{c}_J$ such that

$$\mathbf{x}^{n+1} = T_J\mathbf{x}^n + \mathbf{c}_J.$$

[6 marks]

ii) write out the three equations for $\mathbf{x}^{n+1}$ for the Gauss-Seidel (GS) Method and carry out 2 iterations starting from $\mathbf{x}^0 = \mathbf{0}$. Find $(L + D)^{-1}$ (where L, D are respectively the lower triangular, diagonal parts of A) and hence the iteration matrix T_{GS} and vector $\mathbf{c}_{GS}$ such that

$$\mathbf{x}^{n+1} = T_{GS}\mathbf{x}^n + \mathbf{c}_{GS}.$$

[8 marks]

iii) what necessary and sufficient condition can you use check whether each of the Jacobi and Gauss-Seidel Iteration Methods converges or not? [1 mark]

8. [15 marks]

For the following matrix A

$$\begin{bmatrix} 9 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -14 & 8 \end{bmatrix},$$

given the result

$$(A-6I)^{-1} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -5 & 0 \\ 1 & -14 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & 0.56 & -0.2 \\ 0 & -0.2 & 0 \\ -0.2 & -1.68 & 0.6 \end{bmatrix},$$

use the shifted inverse power method for **two** steps to estimate the eigenvalue near $\gamma = 6$ and its eigenvector. Start from $\mathbf{x}^{(0)} = [-1 \;\; 0 \;\; 1]^T$ and keep at least 4 decimal digits in calculations.

9. [15 marks]

Consider the solution of the following boundary value problem, by the usual finite difference method with $(N+1)\times(M+1) = 3\times 3$ boxes i.e. 4 interior and uniformly distributed mesh points:

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = x - y^2, \qquad p = (x,y) \in \Omega$$

where the domain is the square $\Omega = [0, 0.3]\times[0, 0.3] \in R^2$, with the Dirichlet boundary condition $u|_\Gamma = x + y$ given.

i) Sketch the computational domain and show that the boundary conditions are effectively translated to boundary values of $0.1,\ 0.2$ at the bottom and left sides, and of $0.4,\ 0.5$ at the top and right sides. [3 marks]

ii) Set up the linear system for the four interior unknowns (*without* having to solve it). [12 marks]

10. **[15 marks]**

(a) Compute the Lagrange interpolating polynomial $y = P_3(x)$ of degree 3 that passes through these 4 points (x_j, y_j): [5 marks]

$$(0, \sqrt{17}), \quad (2, \sqrt{5}), \quad (4, 1), \quad (6, \sqrt{5}).$$

(b) The three point quadrature formula can be written as

$$\int_{-1}^{1} f(x)dx = w_0\, f(x_0) + w_1\, f(x_1) + w_2 f(x_2),$$

where $x_0 = -\sqrt{15}/5, \quad x_1 = 0, \quad x_2 = \sqrt{15}/5.$

Find the suitable weights w_j (in exact arithmetic) so that the rule becomes a Gauss type i.e. the rule is exact for degree $0, 1, 2$ polynomials. [6 marks]

Adapt the above rule designed for $[-1, 1]$ to approximate (keeping at least 4 decimal digits in calculations)

$$I = \int_0^1 \frac{10x^4}{\sqrt{100x^5 + 1}} dx$$

and compute the absolute error of the approximation to the true value:

$$I = \frac{\sqrt{101} - 1}{25}.$$

[4 marks]

11. **[15 marks]**

(a) State both the explicit and implicit Euler methods for solving [2 marks]

$$\frac{dy}{dx} = f(x, y), \quad x \geq x_0, \qquad y(x_0) = y_0.$$

Use the explicit Euler method to solve the initial value problem [5 marks]

$$\frac{dy}{dx} = e^{-x}y - 2y + x, \qquad y(0) = 1$$

to obtain $y(0.5)$ with the step length $h = 0.25$.

(b) Combine the nonlinear Newton-Raphson method with the implicit Euler method to solve

$$\frac{dy}{dx} = e^{-x}y - 2y^2 + x, \qquad y(0) = 1$$

for $y(0.5)$ with the step length $h = 0.25$.

(Use no more than 3 iterations in each Newton-Raphson step.) [8 marks]
