1. Let X be the uniform discrete RV taking values 0, 1, and 2 equiprobably, and $Y = X + 1 \pmod{3}$, i.e. Y = X + 1 if X < 2 and Y = 0 if X = 2.

Two players A and B pay the entrance fees $\pounds a$ and $\pounds b$ correspondingly and play the following game: if X > Y then A receives all the money (a + b pounds); if X < Y then B receives all the money. Let a = 1.

(a) Calculate the fair value of b, such that the entrance fee equals expected gain of each participant.

(b) A modification of this game is as follows. The random pairs (X, Y) described above are realized in n = 5 independent rounds, and A receives (a + b) pounds if he wins 3 or more rounds; similarly B receives (a + b) pounds if he wins 3 or more rounds. Calculate the fair value of b, if a = 1, as previously.

(c) Let a = 1 and b be as in item (b). The game described in (b) began, A won the first two rounds, and the players had to stop playing, before the game actually finished. What is the fair division of the prize (a + b)?

2. (a) Let X and Y be two independent Poisson random variables with the same parameter $\lambda > 0$. Find the Probability Mass Function for Z = X + Y.

(b) Let X be the number of particles registered by a Geiger counter during 1/2 hour. It is known that E[X] = 1000. Suggest a proper probability distribution for X.

(c) Using the Central Limit Theorem, evaluate $P\{900 < X < 1000\}$, where X is the random variable described in (b).

3. Suppose X is a continuous random variable. State the definition of the Cumulative Distribution Function and the density function and the relationship between them (no more than 80 words).

Suppose X is a continuous random variable with density

$$f(x) = \frac{1}{2}\sin(x), \quad 0 < x < \pi.$$

(a) Find the Cumulative Distribution Function of X.

(b) Find the density and the Cumulative Distribution Function of $Y = \sqrt{X}$. What is the range of Y?

4. In a physical experiment particles hit a plane at random. The coordinates X and Y of the impact points can be viewed as independent exponential random variables defined by

$$P(X > x) = e^{-\lambda x}, \ x \ge 0, \quad P(Y > y) = e^{-\mu y}, \ y \ge 0.$$

(a) Find the marginal densities of X and Y and the joint density of X and Y.

(b) Consider the polar coordinates

$$R = \sqrt{X^2 + Y^2}, \quad \Theta = \tan^{-1}\left(\frac{Y}{X}\right).$$

Find the joint density of R and Θ . Indicate the range of R and Θ .

(c) Find the marginal density of Θ .

You can use without verification the following formula which is valid for any $\alpha > 0$

$$\int_0^\infty r e^{-r\alpha} dr = \alpha^{-2}$$

5. Suppose X is a discrete random variable with Probability Mass Function

$$P(X = k) = p(k), \quad k = 0, 1, 2, \dots$$

- (a) State the definition of the Moment Generating Function of X.
- (b) Consider a discrete random variable Y with Probability Mass Function

$$p(k) = \left(\frac{1}{2}\right)^{k+1}, \quad k = 0, 1, 2, \dots$$

Find $M_Y(t)$, the Moment Generating Function of Y. Indicate the range of values of t for which $M_Y(t)$ exists.

(c) Using the Moment Generating Function find E[Y] and $E[Y^2]$.

6. The random variables X_1, \ldots, X_n are independent and identically distributed, each with mean $\mu = 1$ and variance $\sigma^2 = 4$.

(a) For the case n = 200 use the Central Limit Theorem to approximate the probability

$$P\{\ln(\bar{X}) > 0\},\$$

where

$$\bar{X} = \frac{1}{200} \sum_{i=1}^{200} X_i$$

is the sample mean and \ln denotes the natural logarithm (i.e. logarithm with base e).

(b) Find the minimum n for which

$$P\{\sum_{i=1}^{n} X_i > 190\} > 0.99.$$

7. (a) State the definition of a chi-square distribution in terms of standard normal random variables.

(b) Suppose U and V are independent chi-square random variables with m and n degrees of freedom, respectively. Show that the random variable W defined by

$$W = \frac{U/m}{V/n}$$

has density

$$f_W(w) = \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \left(\frac{m}{n}\right)^{m/2} w^{\frac{m}{2}-1} \left(1 + \frac{m}{n}w\right)^{-\frac{m+n}{2}}, \quad w > 0.$$

Hint: Recall that the density of a chi-square distribution with n degrees of freedom is

$$f_V(v) = \frac{(1/2)^{n/2}}{\Gamma(\frac{n}{2})} v^{\frac{n}{2}-1} e^{-v/2}$$

Recall also that if A and B are independent random variables, then the density of the quotient C = B/A is

$$f_C(c) = \int |a| f_A(a) f_B(ca) da$$

Finally, recall that the gamma density is

$$f_{\alpha,\lambda}(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \ x > 0, \qquad \alpha, \lambda > 0.$$

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[NORMAL DISTRIBUTION TABLE]