1. (a) Let X be binomially distributed with parameters (n, p). Suppose that $n \to \infty$ and $p \to 0$ in such a way that $np \to \lambda$, some $\lambda > 0$. Show that the distribution of X converges to the Poisson distribution with parameter λ.

[You may use without proof the result that for $-\infty < \mu < \infty$,

$$\lim_{n\to\infty} \left(1 + \frac{\mu}{n}\right)^n = e^{\mu}. \quad]$$

[8 marks]

(b) Write down conditions on n and p under which the Poisson distribution with parameter $\lambda = np$ provides a good approximation to the binomial distribution with parameters (n, p). [2 marks]

(c) In a given population, on average one in five thousand has a rare inherited disease. A random sample of 400 people is taken from this population. It may be assumed that the disease status of any one person in the sample is independent of that of the others. What is the *exact* distribution of the number of people with the disease in the sample? Justify your answer.

Use a Poisson approximation to find the approximate probability that the number of people with the disease in the sample is at most four. Explain why it is appropriate to use the Poisson approximation in this case. [10 marks]

2. Two red balls and two white balls are placed in a bag. The four balls are drawn out one by one, at random and without replacement. The random variable X is the number of white balls drawn before the first red ball is drawn.

   (a) Find the probabilities $P(X = x)$ for $x = 0, 1, 2$. [5 marks]

   (b) Find $E[X]$ and $\mathrm{Var}[X]$. [3 marks]

   (c) Suppose $X_1, X_2, \ldots, X_{100}$ are independent, identically distributed random variables each having the same distribution as the random variable X above. Using the Central Limit Theorem, find an approximation for $P\left(64 \leq \sum_{i=1}^{100} X_i \leq 81\right)$. [8 marks]

   (d) Suppose now that the balls are drawn out of the bag **with** replacement. Let Y denote the number of white balls drawn before the first red ball is drawn in this case. Give an expression for the probability $P(Y = y)$ for $y = 0, 1, 2, \ldots$, justifying your answer. [4 marks]

3. (a) Suppose the random variable T is exponentially distributed with parameter λ, so that T has probability density function

$$f_T(t) \quad = \quad \begin{cases} 0 & t < 0, \\ \lambda \mathrm{e}^{-\lambda t} & t \geq 0. \end{cases}$$

Derive expressions for the (cumulative) distribution function $F_T(t)$ of T and the expectation $E[T]$. [4 marks]

(b) Suppose that T is exponentially distributed with parameter $\lambda = 4$, and that Y is defined by

$$Y \quad = \quad 1 - \mathrm{e}^{-2T}$$

(i) Determine the range of Y. [2 marks]

(ii) Find the (cumulative) distribution function of Y. [8 marks]

(iii) Show that the probability density function of Y is given (within the range of non-zero density) by

$$f_Y(y) \quad = \quad 2(1 - y).$$

[2 marks]

(iv) Find $E[Y]$ and $\mathrm{Var}[Y]$. [4 marks]

4. (a) Give formulae defining the *covariance* and the *correlation* of two random variables X and Y. Explain how correlation values may be interpreted. [6 marks]

(b) Suppose X and Y are discrete random variables with the joint probability mass function given in the following table.

	Y=0	Y=1	Y=2
X=2	0.1	0.1	0.3
X=4	0.2	0.2	0.1

(i) Find the marginal probability mass functions of X and Y, and hence find $E[X]$, $E[Y]$, $\mathrm{Var}[X]$, $\mathrm{Var}[Y]$. [6 marks]

(ii) Find the covariance $\mathrm{Cov}[X, Y]$ and the correlation $\mathrm{Corr}[X, Y]$. [6 marks]

(iii) Comment on your computed correlation value. [2 marks]

5. Suppose that X, Y are continuous random variables with joint density function

$$f_{X,Y}(x,y) \quad = \quad k\left(x(1-x) + y(1-y)\right), \qquad 0 \leq x \leq 1,\ 0 \leq y \leq 1,$$

for some constant k.

(a) Find the value of k, and the marginal densities $f_X(x)$ and $f_Y(y)$. [8 marks]

(b) Let random variables U, V be defined by

$$U = X - Y, \qquad V = X + Y.$$

(i) Show that the Jacobian, J, is given by

$$J \quad = \quad \frac{\partial(x,y)}{\partial(u,v)} \quad = \quad \frac{1}{2}.$$

[3 marks]

(ii) Fnd the joint density $f_{U,V}(u,v)$. [6 marks]

(iii) Sketch the region where the random vector (U, V) has positive density. [3 marks]

6. (a) For a discrete random variable U, the Probability Generating Function $G_U(s)$ of U is defined by $G_U(s) = E\left[s^U\right]$.

Show that (i) $E[U] = G'_U(1)$ and (ii) $\text{Var}[U] = G''(1) + G'(1) - (G'_U(1))^2$, where $G'_U(s)$ and $G''(s)$ denote the first and second derivatives, respectively, of $G_U(s)$ with respect to s. [7 marks]

(b) Suppose V is a discrete random variable with probability mass function

$$\Pr(V = k) \quad = \quad \frac{1}{2^k\ k\ \ln(2)} \qquad k = 1, 2, \ldots.$$

Show that the Probability Generating Function $G_V(s)$ of V is given by

$$G_V(s) \quad = \quad 1 - \frac{\ln(2-s)}{\ln(2)} \qquad \text{for } -2 \leq s < 2.$$

[You may use without proof the result that for $-1 \leq x < 1$,

$$\sum_{i=1}^{\infty} \frac{x^i}{i} = \ln\left(\frac{1}{1-x}\right).$$
]

[7 marks]

(c) Using the probability generating function $G_V(s)$ from part (b) above, or otherwise, find the expectation $E[V]$ and the variance $\text{Var}[V]$. [6 marks]

7. (a) Let Z be a standard normal random variable, so that $Y = Z^2$ follows the chi-squared distribution with 1 degree of freedom. Show that the probability density function $f_Y(y)$ of Y is given by

$$f_Y(y) \quad = \quad \frac{1}{\sqrt{2\pi y}} \exp(-y/2) \qquad y \geq 0.$$

[Recall that the standard normal random variable Z has probability density function $f_Z(z) = \dfrac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right)$ for $-\infty < z < \infty$.] [10 marks]

(b) Given that the chi-squared distribution with 1 degree of freedom is identical to the gamma distribution with parameters $\left(\frac{1}{2}, \frac{1}{2}\right)$, that is, $\chi_1^2 \sim \Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$, write down an expression for the moment generating function $M_Y(t)$ of the random variable Y of part (a) above.

[You may use without proof the result that for $\lambda, \alpha > 0$ the gamma distributed random variable $U \sim \Gamma(\lambda, \alpha)$ has moment generating function given by

$$M_U(t) \quad = \quad \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}. \quad]$$

[2 marks]

(c) State the definition of the chi-squared distribution with n degrees of freedom for $n \geq 1$.

Hence, denoting by V a random variable following the chi-squared distribution with n degrees of freedom, derive an expression for the moment generating function of V.

Given that the chi-squared distribution with n degrees of freedom is identical to the gamma distribution with parameters λ_n, α_n for some $\lambda_n, \alpha_n > 0$, give expressions for λ_n and α_n in terms of n. [8 marks]