PAPER CODE NO.
MATH262

THE UNIVERSITY
of LIVERPOOL

SUMMER 2007 EXAMINATIONS

Bachelor of Arts: Year 2
Bachelor of Science : Year 2
Bachelor of Science : Year 3
Master of Mathematics : Year 2
No qualification aimed for: Year 1

INTRODUCTION TO FINANCIAL MATHEMATICS

TIME ALLOWED : $2\frac{1}{2}$ Hours

INSTRUCTIONS TO CANDIDATES

You may attempt all questions. All answers to section A and the best three answers from section B will be taken into account. The marks noted indicate the relative weight of the questions. Note that all interest rates quoted are per annum; and unless stated otherwise, assume that all interest rates and dividend rates are continuously compounded. Tables of the cumulative normal distribution ($N(x) \equiv \Phi(x)$) are appended.

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SECTION A

1. On a certain date shares in a certain company are priced at 400p and there are in total three million shares. What is the market capitalisation of the company in pounds?

   The company declares a dividend of 2% of the current share price. What will the share price become on the ex-dividend date?

   [**3 marks**]

2. Draw a graph of the value at the expiry date versus share price S at that date for the following portfolios:

   (i) A share and a shorted (i.e. written) call option with strike price 300p for the same share.

   (ii) A call option with strike price 200p and a put option with strike price 300p, both for the same underlying share.

   (iii) A share and a put option with strike price 100p, for the same underlying share.

   [**12 marks**]

3. (i) On a certain date the FTSE100 index is at 6500 and the quoted price of a contract to buy the index 2 years in the future is 7000. If the risk free rate of interest is 5% (compounded continuously), explain how to make a risk free profit (neglecting dividends).

   (ii) Calculate the correct price (to 2 decimal places) for such a contract if the index pays a (compounded continuously) dividend of 2%.

   [**6 marks**]

4. Using arbitrage arguments, derive the relationship between the share price S, the call option price C and the put option price P when the options have a common expiry date at time T in the future and a common strike price E. You may neglect dividends and assume that the risk-free rate of interest is r.

   Using this relationship or otherwise prove that C satisfies the inequality

$$C \geq S - Ee^{-rT},$$

   and derive a similar inequality for P.

   [**10 marks**]

5. Consider a model of share price behaviour where a share valued at 500p may either increase to 540p over 1 month or decrease in value to 460p over the same period. Use this model, neglecting the cost of borrowing money, to value a call option with strike price 520p and expiry date in 1 month. Explain the assumptions you are making.

   [**8 marks**]

6. If a random variable S is distributed according to a log-normal distribution (with $\ln S$ having mean $\bar{l} = \ln 500$ and variance $\sigma^2 = 0.07$), find the probability (to 3 significant figures) that $S > 600$. You may use tables of N (with interpolation where necessary).

   [**8 marks**]

7. The explicit solution of the Black Scholes equation for the price of a call option $C(S(t),t)$ at time t with expiry at time T_E, strike price E and current share price S with risk free rate of interest r and volatility σ is:

$$C(S,t) = SN(d_1) - Ee^{-rT}N(d_2)$$

with $T = T_E - t$, $\sigma\sqrt{T}d_1 = \ln(S/E) + (r + \sigma^2/2)T$ and $\sigma\sqrt{T}d_2 = \ln(S/E) + (r - \sigma^2/2)T$.

Use this explicit solution of the Black-Scholes equation to evaluate (using tables for N with interpolation where necessary) the price of a call option (to 1 decimal place) with expiry in 3 months, strike price 700p and current share price 650p. You may assume that the risk free rate of interest is 5.25% and the volatility is 0.4 (in time units of years).

[**8 marks**]

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SECTION B

8. For consecutive weeks, the price S of a share was as follows (in pence):

   700 710 703 693 695 681 690 687 675 680.

   Assuming this share follows a log-normal random walk, evaluate the mean growth rate μ (of $\ln S$) and volatility σ from these data. Express your results to 3 significant figures using time units of years.

   With these same assumptions, derive an expression (as an integral which you need not evaluate) for the expected value of a call option with strike price 680p and expiry in 3 months when the current share price is 675p.

   [15 marks]

9. Consider a model of share price behaviour where the share price S may either increase by a factor 8/7 or decrease by a factor of 7/8 in each 1 month period.

   Use this model to value (to the nearest penny) a call option with strike price 3000p and expiry in two months when the current share price is 3136p. You may neglect the cost of borrowing.

   Explain what is meant by the term "active hedging" in the context of this example.

   [15 marks]

10. Consider a continuous random walk in the share price

$$dS = \sigma S dX + \mu S dt. \qquad (A)$$

Explain the meaning of the symbols in this expression.

Show that if S satisfies Eq. (A), then a smooth function $V(S,t)$ satisfies the equation

$$dV = \sigma S \frac{\partial V}{\partial S} dX + \left[\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right] dt$$

explaining carefully any assumptions you make. Using this result, ignoring dividend payments, and assuming a risk-free (continuously compounded) rate of interest r, present the derivation of the Black-Scholes differential equation satisfied by an option $V(S,t)$ associated with an underlying asset S.

Explain briefly why the price of a forward contract does not satisfy the Black-Scholes equation.

[15 marks]

11. Consider the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} = 0. \qquad (1)$$

Show that the substitution $S = Ee^x$ (where E is a constant) reduces Eq. (1) to the equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \left[\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x} \right] = 0. \qquad (2)$$

Show that the further substitution $\tau = \frac{1}{2}(T_E - t)$ (where T_E is a constant) reduces Eq. (2) to the equation

$$\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x} \qquad (3)$$

and that the final substitution $V = u(x,\tau)e^{x/2-\tau/4}$ reduces Eq. (3) to the standard diffusion equation for $u(x,\tau)$.

[15 marks]