MATH262 May 2006
time: 2.5 hours

Instructions to candidates

You may attempt all questions

All answers to section A and the best three answers from section B will be taken into account.

The marks noted indicate the relative weight of the questions.

Note that all interest rates quoted are per annum; and unless stated otherwise, assume that all interest rates and dividend rates are continuously compounded. Tables of the cumulative normal distribution ($N(x) \equiv \Phi(x)$) are appended.

SECTION A

1. Draw a graph of the value at the expiry date versus share price S at that date for the following portfolios:

(i) A share and a shorted (i.e. written) put option with strike price 100p, both for the same underlying share.

(ii) A put option with strike price 200p and a call option with strike price 300p, both for the same underlying share.

[8 marks]

2. Explain carefully using arbitrage arguments why the price for a future contract on an asset paying no dividend is given by

$$F = Se^{rt},$$

where S is the asset price when the contract is made, t is the time between when the contract is made and the time it is fulfilled and the risk-free rate of interest is r.

Indicate briefly how the result is altered if (a) the asset pays a single dividend D during the contract period, and (b) if the asset pays a continuously compounded dividend with dividend rate d.

If on a certain date the FTSE100 index is at 6000, calculate (to 2 decimal places) the price of a futures contract to deliver this index to a customer 8 months in the future (with payment of F then). You may assume that the continuously compounded dividend paid on the index is 3% and the risk free rate of interest is 4.5%.

[7 marks]

3. On a certain date shares in a certain company are priced at 700p and there are in total five million shares. What is the market capitalisation of the company in pounds?

The company makes a rights issue of three shares for every existing share, priced at 300p. What will the share price be after the rights issue? Explain briefly the consequences for a shareholder of (a) taking up the rights issue (b) not taking up the rights issue. [7 marks]

4. Stating your assumptions, derive the relationship between the share price S, the call option price C and the put option price P when the options have a common expiry date at time T in the future and a common strike price E. You may neglect dividends and assume that the risk-free rate of interest is r.

A call option with expiry in 15 months and strike price 600p is quoted at 140p. The share price is quoted at 550p. Assuming the risk free rate of interest is 4% and neglecting any dividend payments, value a put option (to nearest penny) with the same expiry date and strike price. [8 marks]

5. Consider a model of share price behaviour where a share valued at 100p may either increase to 150p over 1 month or decrease in value to 50p over the same period. Use this model, neglecting the cost of borrowing money, to value a call option with strike price 120p and expiry date in 1 month. Explain the assumptions you are making.

Also value a put option with the same strike price and expiry date.

[10 marks]

6. A random variable X has a probability density function given by

$$f(x) = ae^{-x} \quad \text{when} \quad x \geq 0$$

and

$$f(x) = 0 \quad \text{when} \quad x < 0,$$

where a is a constant.

Calculate a, $E(X)$ and $E(X^2)$. [6 marks]

7. The standard diffusion equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Consider the partial differential equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}.$$

Use the substitution $v = e^{\gamma x + \beta t} u(x, t)$ to reduce this to the standard diffusion equation for $u(x, t)$, and find the constants γ and β.

[9 marks]

SECTION B

8. For consecutive months, the price S of a share was as follows (in pence):

600 615 592 598 623 686 662 676 690.

Assuming this share follows a log-normal random walk, evaluate the mean growth rate μ (of $\ln S$) and volatility σ from these data. Express your results to 3 significant figures using time units of years.

With these same assumptions, write down and evaluate (using tables for $N(x)$; you need not use interpolation) an integral which gives the probability that the value of a share will be less than 750 three months after it is 690. Give your result to 2 significant figures.

[15 marks]

9. Consider a model of share price behaviour where the share price S may either increase or decrease by a factor of 5/4 or 4/5 in each 1 month period.

Use this model to value (to the nearest penny) a put option with strike price 6000p and expiry in two months when the current share price is 5600p. You may neglect the cost of borrowing.

[15 marks]

10. The explicit solution of the Black Scholes equation for the price of a call option $C(S(t), t)$ at time t with expiry at time T_E, strike price E and current share price S with risk free rate of interest r and volatility σ is:

$$C(S,t) = SN(d_1) - Ee^{-rT}N(d_2)$$

with $T = T_E - t$, $\sigma\sqrt{T}d_1 = \ln(S/E) + (r + \sigma^2/2)T$ and $\sigma\sqrt{T}d_2 = \ln(S/E) + (r - \sigma^2/2)T$. What assumption concerning dividends on the share is required to obtain the above solution?

Explain the significance of the function $N(x)$, and give the values of $N(\infty)$ and $N(-\infty)$.

$C(S(t), t)$ satisfies definite boundary conditions if (i) $t = T_E$, (ii) $S = 0$, (iii)$S \to \infty$. Write down these boundary conditions and explain carefully (using arbitrage arguments where appropriate) why they must be satisfied. Verify that the explicit solution given above satisfies all the boundary conditions.

[15 marks]

11. Consider a continuous random walk in the share price

$$dS = \sigma S dX + (\mu - d) S dt. \tag{1}$$

Explain the meaning of the symbols in this expression.

Show that if S satisfies Eq. (1), then a smooth function $V(S,t)$ satisfies the equation

$$dV = \sigma S \frac{\partial V}{\partial S} dX + \left[(\mu - d) S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right] dt$$

explaining carefully any assumptions you make. Using this result, present the derivation of the Black-Scholes differential equation satisfied by an option $V(S,t)$ for the case when the underlying asset S pays a continuously compounded dividend with dividend rate d and the risk-free rate of interest is r.

[15 marks]