MATH262

Instructions to candidates

You may attempt all questions

All answers to section A and the best three answers from section B will be taken into account.

The marks noted indicate the relative weight of the questions.

Note that all interest rates quoted are per annum and that tables of the cumulative normal distribution ($N(x) \equiv \Phi(x)$) are appended.

SECTION A

1. Draw a graph of the value at the expiry date versus share price S at that date for the following portfolios:

(i) A share and a put option with strike price 500p.

(ii) A call option with strike price 500p and a shorted (ie sold short or written) put option with strike price 300p.

[6 marks]

2. On a certain date when the FTSE100 index is at 5500, a broker writes a futures contract at price F to deliver this index to a customer 6 months in the future (with payment of F then). Describe the risk-free strategy available to the broker to provide this future contract and hence obtain the price F (to 2 d.p.) of the future contract. You may assume that the dividend paid on the index is 2% and the risk free rate of interest is 4%.

[8 marks]

3. Stating your assumptions, derive the relationship between the share price S, the call option price C and the put option price P when the options have a common expiry date at time t in the future and a common strike price E. You may neglect dividends and assume that the risk-free rate of interest is r.

A call option with expiry in 4 months and strike price 500p is quoted at 126p. The share price is quoted at 507p. Assuming the risk free rate of interest is 4% and neglecting any dividend payments, value a put option with the same expiry date and strike price. [8 marks]

4. Consider a model of share price behaviour where a share valued at 500p may increase to 600p over 2 months with probability p and may decrease in value to 400p over 2 months with probability $(1-p)$. Use this model, neglecting the cost of borrowing money, to value a call option with strike price 450p and expiry date in 2 months. Explain the assumptions you are making.

Also value (to 2 d.p.) the same call option with the same strike price and expiry date taking account of the cost of borrowing at the risk free rate of 4%.

[9 marks]

5. If a random variable S is distributed according to a log-normal distribution (with $\ln S$ having mean $\bar{l} = \ln 500$ and variance $\sigma^2 = 0.16$), find the probability (to 3 significant figures) that $S < 400$. You may use tables of N, with interpolation where necessary. [8 marks]

6. If S is a random variable satisfying the stochastic differential equation

$$dS = S\sigma dX + S\mu dt$$

where the expectation values are given by $E(dX) = 0$ and $E(dX^2) = dt$, use Ito's lemma to find the corresponding equation for the random variable $g(S) = \ln S$. [6 marks]

7. In this question you may assume the explicit solution of the Black Scholes equation for the price of a call option at time t with expiry at time T_E, strike price E and current share price S with risk free rate of interest r and volatility σ is:

$$C(S,t) = SN(d_1) - Ee^{-rT}N(d_2)$$

with $T = T_E - t$, $\sigma\sqrt{T}d_1 = \ln(S/E) + (r + \sigma^2/2)T$ and $\sigma\sqrt{T}d_2 = \ln(S/E) + (r - \sigma^2/2)T$

Use this explicit solution of the Black Scholes equation to evaluate (using tables for N with interpolation where necessary) the price of a call option (to 1 decimal place) with expiry in 3 months, strike price 300p and current share price 380p. You may assume that the risk free rate of interest is 4% and the volatility is 0.3 (in time units of years). [10 marks]

SECTION B

8. For consecutive months, the price S of a share was as follows (in pence):

321 335 355 312 270 320 340 376 403 370

Assuming this share follows a log-normal random walk, evaluate the mean growth rate (of $\ln S$) and volatility from this data. Express your results to 3 significant figures using time units of years.

Assuming a random walk in the log of the share price with growth rate (in $\ln S$) of 0.03 and with the value of 0.3 for the volatility (in time units of years), estimate the probability (to a precision of 0.1%) that a share will be worth more than 400p four months later when its current price is 370p. [tables of N may be used] [15 marks]

9. Consider a model of share price behaviour where the share price S may increase or decrease by a factor of $5/4$ or $4/5$ (with probabilities p, $(1-p)$ respectively) in each 2 month period. Use this model to value (to 1 d.p.) a call option with strike price 440p and expiry in four months when the current share price is 400p. You may neglect the cost of borrowing.

For the case when $p = 1/2$, consider this discrete random walk using as a variable $X = \ln S$. Obtain an expression for the variance of X after n random steps assuming $X = x_0 = \ln 400$ initially. Hence estimate the volatility in this model (to 3 significant figures).

[15 marks]

10. Consider a continuous random walk in the share price

$$dS = S\sigma dX + S\mu dt$$

where you should explain the meaning of the symbols in this expression.

Use Ito's Lemma to show that the change in the value $V(S,t)$ of an option is given by

$$dV = \sigma S \frac{\partial V}{\partial S} dX + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})dt$$

State the assumptions used in the derivation of the Black Scholes equation.

Neglecting the cost of borrowing and ignoring dividends, present the derivation of the Black Scholes differential equation satisfied by $V(S,t)$.

Describe briefly the hedging strategy to be used by the writer of a call option.

[15 marks]

11. The standard diffusion equation is

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

(a) Verify that $u(x,\tau) = c\tau^{-1/2} e^{-x^2/(4\tau)}$ with constant c is a solution to the standard diffusion equation.

Obtain an expression (as an integral which you need not evaluate) for the solution for $\tau > 0$ given initial condition that $u(x,0) = \max(x,0)$.

(b) Consider the partial differential equation, where a and b are constant,

$$\frac{\partial v}{\partial t} = a + b\frac{\partial^2 v}{\partial x^2}$$

Make substitutions to reduce this to the standard diffusion equation for $u(x,\tau)$.

[15 marks]