MATH262

Instructions to candidates

You may attempt all questions

All answers to section A and the best three answers from section B will be taken into account.

The marks noted indicate the relative weight of the questions.

Note that all interest rates quoted are per annum and that tables of the cumulative normal distribution ($N(x) \equiv \Phi(x)$) are appended.

SECTION A

1. Draw a graph of the value at the expiry date versus share price S at that date for the following portfolio:

(i) A call option with strike price 500p and a put option with strike price 300p.

(ii) A share and a shorted (ie sold short or written) call option with strike price 500p. [6 marks]

2. On a certain date the FTSE100 index is at 6000 and the price of a futures contract to buy this index 4 months in the future is 6070. You may assume that the dividend paid on the index is 2% and the risk free rate of interest is 5%. Give arguments to show whether it is theoretically possible to make a risk free profit - and if so how. [8 marks]

3. A call option with expiry in 3 months and strike price 800p is quoted at 126p. The share price is quoted at 807p. Assuming the risk free rate of interest is 5% and neglecting any dividend payments, use arbitrage arguments (which you should explain) to value a put option with the same expiry date and strike price. [8 marks]

4. Consider a model of share price behaviour where a share valued at 600p may increase to 700p over 3 months with probability p and may decrease in value to 500p over 3 months with probability $(1-p)$. Use this model, neglecting the cost of borrowing money, to value a call option with strike price 550p and expiry date in 3 month. Explain the method you are using.

Also value the same call option with the same strike price and expiry date taking account of the cost of borrowing at the risk free rate of 5%.

[9 marks]

5. If a random variable S is distributed according to a log-normal distribution (with $\ln S$ having mean $\bar{l} = \ln 1000$ and variance $\sigma^2 = 0.09$), find the probability (to 3 significant figures) that $S > 1100$. You may use tables of N, with interpolation where necessary. [8 marks]

6. If S is a random variable satisfying the stochastic differential equation

$$dS = S\sigma dX + S\mu dt$$

where the expectation values are given by $E(dX) = 0$ and $E(dX^2) = dt$, use Ito's lemma to find the corresponding equation for the random variable $g(S) = S^2$. [6 marks]

7. In this question you may assume the explicit solution of the Black Scholes equation for the price of a call option at time t with expiry at time T_E, exercise price E and current share price S with risk free rate of interest r and volatility σ is:

$$C(S,t) = SN(d_1) - Ee^{-rT}N(d_2)$$

with $T = T_E - t$, $\sigma\sqrt{T}d_1 = \ln(S/E) + (r + \sigma^2/2)T$ and $\sigma\sqrt{T}d_2 = \ln(S/E) + (r - \sigma^2/2)T$

Use this explicit solution of the Black Scholes equation to evaluate (using tables for N with interpolation where necessary) the price of a call option (to 1 decimal place) with expiry in 2 months, exercise price 1400p and current share price 1380p. You may assume that the risk free rate of interest is 5% and the volatility is 0.3 (in time units of years). [10 marks]

SECTION B

8. For consecutive months, the price S of a share was as follows (in pence):

521 535 555 512 470 520 540 576 603 570

Assuming this share follows a log-normal random walk, evaluate the mean growth rate (of $\ln S$) and volatility from this data. Express your results to 3 significant figures using time units of years.

Assuming a random walk in the log of the share price with no growth term (in $\ln S$) and with the value of 0.3 for the volatility in units of years, estimate the probability (to a precision of 0.1%) that a share will be worth more than 600p three months later when its current price is 570p. [tables of N may be used] [15 marks]

9. Consider a model of share price behaviour where the share price S may increase or decrease by a factor of 10/9 or 9/10 (with probabilities p, $(1-p)$ respectively) in each 1 month period. Use this model to value a call option with strike price 290p and expiry in two months when the current share price is 270p. You may neglect the cost of borrowing.

Describe the hedging strategy to be used by the writer of such a call option for the above model of share pricing.

[15 marks]

10. Consider a continuous random walk in the share price

$$dS = S\sigma dX + S\mu dt$$

where you should explain the meaning of the symbols in this expression.

Show that the change in the value $V(S,t)$ of an option is given by

$$dV = \sigma S \frac{\partial V}{\partial S} dX + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})dt$$

Explain the assumptions used in the derivation of the Black Scholes equation.

Assuming the risk free rate of interest is r and ignoring dividends, present the derivation of the Black Scholes differential equation satisfied by $V(S,t)$.

[15 marks]

11. The standard diffusion equation is

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

(a) Verify that $u(x,\tau) = c\tau^{-1/2}e^{-x^2/(4\tau)}$ with constant c is a solution to the standard diffusion equation. Show that u may be considered as a probability density function, and determine c in that case.

(b) Consider the partial differential equation, where a and b are constant,

$$\frac{\partial v}{\partial \tau} = av + b\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}$$

Use the substitution $v = e^{\alpha x + \beta \tau}u$ to reduce this to the standard diffusion equation for $u(x,\tau)$, and express the constants α and β in terms of a and b.

[15 marks]