Instructions to candidates

You may attempt all questions

All answers to section A and the best three answers from section B will be taken into account.

The marks noted indicate the relative weight of the questions.

Note that all interest rates quoted are per annum and that tables of the cumulative normal distribution ($N(x) \equiv \Phi(x)$) are appended.

SECTION A

1. Draw a graph of the value at the expiry date versus share price S at that date for the following portfolio:

(i) A call option with strike price 500 and a shorted (ie sold short) call option with strike price 400.

(ii) A put option with strike price 500 and a shorted (ie sold short) call option with strike price 500. [6 marks]

2. On a certain date the FTSE100 index is at 6000 and the price of a contract to buy this index 3 months in the future is 6035. You may assume that the dividend paid on the index is 2% and the risk free rate of interest is 5%. Give arbitrage arguments to show whether it is theoretically possible to make a risk free profit - and if so how. [8 marks]

3. Consider a model of share price behaviour where a share valued at 500p may increase to 600p over 1 month with probability p and may decrease in value to 400p over 1 month with probability $(1-p)$. Use this model, neglecting the cost of borrowing money, to value a call option with strike price 450p and expiry date in 1 month. Explain the method you are using.

Also value a put option with the same strike price and expiry date.

[9 marks]

4. A call option with expiry in 3 months and strike price 1000p is quoted at 126p. The share price is quoted at 1007p. Assuming the risk free rate of interest is 5% and neglecting any dividend payments, use arbitrage arguments to value a put option with the same expiry date and strike price. [8 marks]

5. If a random variable S is distributed according to a log-normal distribution (with $\ln S$ having mean $\bar{l}$ and variance σ^2), find the expectation value of S^2, namely $E(S^2)$. [8 marks]

6. If S is a random variable satisfying the stochastic differential equation

$$dS = S\sigma dX + S\mu dt$$

where $E(dX) = 0$ and $E(dX^2) = dt$, use Ito's lemma to find the corresponding equation for the random variable $g(S) = \ln S$. [6 marks]

7. In this question you may assume the explicit solution of the Black Scholes equation for the price of a call option at time t with expiry at time T_E, exercise price E and current share price S with risk free rate of interest r and volatility σ is:

$$C(S,t) = SN(d_1) - Ee^{-rT}N(d_2)$$

with $T = T_E - t$, $\sigma\sqrt{T}d_1 = \ln(S/E) + (r + \sigma^2/2)T$ and $\sigma\sqrt{T}d_2 = \ln(S/E) + (r - \sigma^2/2)T$

Use this explicit solution of the Black Scholes equation to evaluate (using tables for N) the price of a call option with expiry in 3 months, exercise price 1200 and current share price 1216. You may assume that the risk free rate of interest is 5% and the volatility is 0.38 (in time units of years). [10 marks]

SECTION B

8. For consecutive weeks at the end of 1999 the price S of BT shares was as follows (in pence):

1085 1107 1254 1229 1305 1392 1415 1316 1446 1513

Assuming this share follows a log-normal random walk, evaluate the mean growth rate (of $\ln S$) and volatility from this data. Express your results using time units of years.

Assuming a random walk in the log of the share price with no growth term (in $\ln S$) and with the value of 0.429 for the volatility in units of years, estimate the probability that a BT share will be worth more than 1600p three months later when its current price is 1513p. [tables of N may be used] [15 marks]

9. Consider a model of share price behaviour where the share price S may increase or decrease by a factor of 10/9 or 9/10 (with probabilities p, $(1-p)$ respectively) in each 1 month period. Use this model to value a call option with strike price 850p and expiry in two months when the current share price is 810p. You may neglect the cost of borrowing.

For the case when $p = 1/2$, consider this discrete random walk using as a variable $X = \ln S$. Obtain an expression for the variance of X after n months assuming $X = x_0 = \ln 810$ initially. Hence estimate the volatility in this model. [15 marks]

10. Consider a continuous random walk in the share price

$$dS = S\sigma dX + S\mu dt$$

where the symbols in this expression have the conventional interpretation.

Show that the change in the value $V(S,t)$ of an option is given by

$$dV = \sigma S \frac{\partial V}{\partial S} dX + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}) dt$$

By constructing a risk free portfolio, derive the Black Scholes differential equation satisfied by $V(S,t)$, assuming the risk free rate of interest is r and ignoring dividends.

Give the boundary conditions on $V(S,t)$ appropriate for a put option with exercise price E and expiry time T.

Consider the value F of a forward contract to deliver a share at time T. Discuss why the Black Scholes differential equation does not apply to this case.

[15 marks]

11. The standard diffusion equation is

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

(a) Consider the partial differential equation, where a and b are constant,

$$\frac{\partial v}{\partial \tau} = av + bs\frac{\partial v}{\partial s} + s^2 \frac{\partial^2 v}{\partial s^2}$$

Use the substitutions $s = e^x$ and $v = e^{\alpha x + \beta \tau} u$ to reduce this to the standard diffusion equation for $u(x,\tau)$, and express the constants α and β in terms of a and b.

(b) Show that $u(x,\tau) = U(\xi)$ where $\xi = x/\sqrt{\tau}$ is a solution to the standard diffusion equation and determine the general expression for $U(\xi)$
[15 marks]