1. (a) Define

 $x_1 =$ Tons of A produced per week

 $x_2 =$ Tons of B produced per week

Problem is to maximise $x_0 = 400x_1 + 300x_2$ (£ per week) subject to

$0.3x_1 + 0.4x_2$	\leq	300 (tons per week)
x_1	\geq	500 (tons per week)
x_2	\geq	200 (tons per week)
x_1, x_2	\geq	0

(b) Define

 $x_1 =$ Number of drivers working morning shift

 $x_2 =$ Number of drivers working split shift

 $x_3 =$ Number of drivers working afternoon shift

Problem is to minimise $x_0 = x_1 + x_2 + x_3$ (drivers) subject to

Redundant constraints: $x_3 \ge 20, x_1 \ge 0, x_3 \ge 0.$

Evaluating z(x, y) at vertices of feasible region, z(2, 0) = 2, z(4, 0) = 4, z(0, 3) = 12, z(0, 2) = 8, so maximum is z = 12 at (x, y) = (0, 3).

At optimality, constraints $x \ge 0$ and $3x + 4y \le 12$ are binding; $y \ge 0$, $x + y \ge 2$ and $3x + 2y \le 12$ are non-binding.

Constraint $3x + 2y \le 12$ is redundant.

(b) Introducing slack variables s_1, s_2, s_3 , then tableaux are

					a	r_1	x_2	x_3	s_1	s_2	s_3						
				-	x_0 –	-3	-1	-1	0	0) 0	0					
				-	s_1	1	1	1	1	0) 0	4					
					s_2	2	1	0	0	1	0	2					
				_	s_3 –	-1	1	1	0	0) 1	2					
	x_1	x_2	x_3	s_1	s_2	s_3					x_1	x_2	x_3	s_1	s_2	s_3	
x_0	0	0.5	-1	0	1.5	0	3			x_0	0	0.5	-1	0	1.5	0	3
s_1	0	0.5	1	1	-0.5	0	3	or		s_1	0	0.5	1	1	-0.5	0	3
x_1	1	0.5	0	0	0.5	0	1			x_1	1	0.5	0	0	0.5	0	1
s_3	0	1.5	1	0	0.5	1	3			s_3	0	1.5	1	0	0.5	1	3
	$ _{r_1}$	r_{0}	r_{2}	S1	S 0	Sa	.		_		$ _{r_1}$	ro	r_{2}	S 1	80	80	
	x_1		x_3	s_1	s_2			-			x_1	$\frac{x_2}{2}$	x_3	s_1	$\frac{s_2}{2}$	$\frac{s_3}{1}$	6
$\frac{x_0}{x_2}$	0	1	0	1	1	0) 6	-		x ₀	0	2	0	0	2	1	6
$\overline{x_3}$	0	0.5	0	1	1 - 0.5	0) 6) 3	-	_	s_1	0	2 - 1	0	0	$\frac{2}{2}$ -1	1 -1	0
x_3 x_1		$ \begin{array}{c} 1 \\ 0.5 \\ 0.5 \end{array} $	0 1 0	1 1 0	-0.5 0.5	000000000000000000000000000000000000000) 6) 3) 1	-	_	$\frac{s_1}{x_1}$	0	2 -1 0.5	0 0 0	0 1 0	2 -1 0.5	$\frac{1}{-1}$ 0	0 1
x_3 x_1 s_3	0 0 1 0	$ \begin{array}{c} 1 \\ 0.5 \\ 0.5 \\ 1 \end{array} $	0 1 0 0		1 - 0.5 0.5 1	0 0 0 1) 6) 3) 1 . 0	- - -		s_1 x_1 x_3	0 0 1 0	$2 \\ -1 \\ 0.5 \\ 1.5$	0 0 0	0	$\frac{2}{2}$ -1	1 -1	0
	0 0 1 0 ptim	0.5 0.5 1 al sol	0 1 0 0 ution	$ \begin{array}{r} 1 \\ 1 \\ 0 \\ -1 \\ 1 is x $	$ \begin{array}{r} 1 \\ -0.5 \\ 0.5 \\ 1 \\ 0 = 6, \end{array} $	0 0 0 1 whe	$\begin{array}{c c} 0 & 6 \\ 0 & 3 \\ 0 & 1 \\ 0 \\ en x_1 \end{array}$			$ \begin{array}{r} s_1\\ x_1\\ x_3\\ = 0 \end{array} $	$ \begin{array}{c c} 0 \\ 0 \\ $	2 -1 0.5 1.5 = 3.	0 0 0 1	0 1 0	2 -1 0.5	$\frac{1}{-1}$ 0	0 1
	0 0 1 0 ptim	$ \begin{array}{c} 1 \\ 0.5 \\ 0.5 \\ 1 \end{array} $	0 1 0 0 ution	$ \begin{array}{r} 1 \\ 1 \\ 0 \\ -1 \\ 1 is x $	$ \begin{array}{r} 1 \\ -0.5 \\ 0.5 \\ 1 \\ 0 = 6, \\ + x_2 - 4 \\ \hline $	$\begin{array}{c} 0\\ 0\\ 0\\ 1\\ whe\\ +x_3 \end{array}$	$\begin{array}{c ccc} 0 & 6 \\ 0 & 3 \\ 0 & 1 \\ 0 & 0 \\ \end{array}$	1 +	- 0 -		$ \begin{array}{c} 0 \\ 0 \\ $	$ \begin{array}{r} 2 \\ \hline 2 \\ $	0 0 0 1	0 1 0	2 -1 0.5	$\frac{1}{-1}$ 0	0 1
	0 0 1 0 ptim	0.5 0.5 1 al sol	0 1 0 0 ution	$ \frac{1}{1} $ 0 $ -1 $ n is x_1	$ \begin{array}{r} 1 \\ -0.5 \\ 0.5 \\ 1 \\ 0 = 6, \\ + x_2 - \\ 2x_1 - \\ 2x_1 - \\ \end{array} $	$\begin{array}{c} 0\\ 0\\ 0\\ 1\\ whe\\ +x_3\\ +x_2 \end{array}$	$\begin{array}{c c} 0 & 6 \\ 0 & 3 \\ 0 & 1 \\ 0 \\ en x_1 \\ = \\ = \end{array}$	1 + 2 + 2	- 0 - - 0	$ \begin{array}{r} s_1 \\ x_1 \\ x_3 \\ \hline = 0 \\ + 3 \\ \hline = \end{array} $	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	$ \begin{array}{r} 2 \\ \hline $	0 0 0 1 4	0 1 0 0	2 -1 0.5	$\frac{1}{-1}$ 0	$\begin{array}{c} 0 \\ 1 \end{array}$
	0 0 1 0 ptim	0.5 0.5 1 al sol	0 1 0 0 ution	$ \frac{1}{1} $ 0 $ -1 $ n is x_1	$ \begin{array}{r} 1 \\ -0.5 \\ 0.5 \\ 1 \\ 0 = 6, \\ + x_2 - 4 \\ \hline $	$\begin{array}{r} 0\\ 0\\ 0\\ 0\\ 1\\ whee \\ +x_3\\ +x_2\\ +x_3 \end{array}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	1 + 2 + -1	- 0 - - 0	$ \begin{array}{r} s_1 \\ x_1 \\ x_3 \\ \hline = 0 \\ + 3 \\ \hline = \end{array} $	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	$ \begin{array}{r} 2 \\ \hline $	0 0 0 1 4	0 1 0 0	2 -1 0.5	$\frac{1}{-1}$ 0	$\begin{array}{c} 0 \\ 1 \end{array}$

Basic variables are x_1, x_3, s_3 ; alternative optimal basis x_1, x_3, s_1 . Or vice-versa. Or could give either x_1, x_3, s_2 or x_1, x_2, x_3 as alternative basis.

Evaluating z at vertices gives z(0,0) = 0, z(2,0) = 4, z(0.5,3) = 7, z(0,2) = 4 so maximal value is z = 7 at (x, y) = (0.5, 3).

- (i) Lines -2x + y = 2 and 2x = 4 intersect at (2, 6), constraint $2x + y \le c$ become redundant when line 2x + y = c passes through the same point, so when $c = 2 \times 2 + 6 = 10$.
- (ii) Optimum remains at (0.5, 3) until objective line 2x + by = const is parallel to the line 2x + y = 4, that is when b = 1, so within the range $b \ge 1$ the optimum point remains the same.
- (iii) Optimal solution is affected when k increases so that the line 2x + ky = 4 crosses into the feasible region, which happens when it coincides with the line 2x + y = 4, so when k = 1. That is, k can increase by 1 before solution is affected.
- (b) Dual simplex method appropriate when there are \geq constraints, and usually for minimisation problems.

Introducing surplus variables s_1, s_2 and slack variable s_3 , then tableaux are

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	3	1	1	0	0	0	0
s_1	1	-2	-1	1	0	0	-2
s_2	-2	1	-1	0	1	0	-5
s_3	1	1	2	0	0	1	12
	1						1
	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	$\begin{array}{c} x_1 \\ 1 \end{array}$	$\frac{x_2}{2}$	$\frac{x_3}{0}$	$\frac{s_1}{0}$	$\frac{s_2}{1}$	$\frac{s_3}{0}$	-5
$\frac{x_0}{s_1}$			-				$-5 \\ 3$
	1	2	0	0	1	0	Ů

So optimal solution is $x_0 = -5$, when $x_1 = 0$, $x_2 = 0$, $x_3 = 5$.

D: minimise $y_0 = 7y_1 + y_2 + 2y_3$ subject to

Given $x_1^* = 2, x_2^* = 1$ then

 $2x_1^* + 3x_2^* = 7$, first constraint satisfied with equality; $-x_1^* + x_2^* = -1$, second constraint not satisfied with equality; $x_1^* = 2$, third constraint satisfied with equality.

Given $y_1^* = \frac{1}{3}, y_2^* = 0, y_3^* = \frac{1}{3}$ then

 $2y_1^* - y_2^* + y_3^* = 1$, first constraint satisfied with equality;

 $3y_1^* + y_2^* = 1$, second constraint satisfied with equality.

Complementary slackness requires that for those constraints not satisfied with equality, the dual variable is zero. For the primal, the second constraint is the only one not satisfied with equality, so since $y_2^* = 0$, complementary slackness is satisfied. For the dual, both constraints are satisfied with equality, so complementary slackness is satisfied.

(b) O is inferior to A, B, C, D; D is inferior to B, C.

The NIS consists of the edges AB and BC.

When w = 0, optimal point is A.

When w = 1, optimal point is C.

Optimum moves from A to B when

$$19(1 - w) + 2w = 14(1 - w) + 16w$$

$$19 - 17w = 14 + 2w$$

$$-19w = -5$$

$$w = 5/19$$

Optimum moves from B to C when

$$14(1 - w) + 16w = 10(1 - w) + 17w$$

$$14 + 2w = 10 + 7w$$

$$-5w = -4$$

$$w = 4/5$$

 $0 \le w < 5/19$: point A optimal w = 5/19: edge AB optimal 5/19 < w < 4/5: point B optimal w = 4/5: edge BC optimal $4/5 < w \le 1$: point C optimal require

$$L_x = 2x + 3y + 5 + 4\lambda = 0$$

$$L_y = 2y + 3x + 10 + \lambda = 0$$

$$L_\lambda = 4x + y - 5 = 0$$

To solve for x, y, λ ,

$$3L_x - 2L_y = 5y - 5 + 10\lambda = 0 \Rightarrow y = 1 - 2\lambda$$

then $L_x = 0 \Rightarrow x = -(3y + 5 + 4\lambda)/2 = -(3 - 6\lambda + 5 + 4\lambda)/2 = -(8 - 2\lambda)/2 = -4 + \lambda$
next, $L_\lambda = 0 \Rightarrow 4(-4 + \lambda) + (1 - 2\lambda) - 5 = 0 \Rightarrow 2\lambda - 20 = 0 \Rightarrow \lambda = 10$
hence minimum occurs at $x^* = 6, y^* = -19$.

(b)

$$f(x,y) = x^{2} + y^{2} - xy - x + 4$$

$$\nabla f = (2x - y - 1, 2y - x)$$

Starting from $(x_0, y_0) = (0, 0)$, have $(\nabla f)_0 = (-1, 0)$, so search along the line

$$\begin{array}{rcl} (x,y) &=& (0,0)+\theta(-1,0) &=& (-\theta,0)\\ \\ \mathrm{so}\ f(x,y) &=& \theta^2+\theta+4\\ \\ df/d\theta &=& 2\theta+1 \end{array}$$

stationary point at $\theta = -0.5$, so $(x_1, y_1) = (0.5, 0)$. Hence $(\nabla f)_1 = (0, -0.5)$, so search along the line

$$(x, y) = (0.5, 0) + \theta(0, -0.5) = (0.5, -0.5\theta)$$

so $f(x, y) = 0.25 + 0.25\theta^2 + 0.25\theta - 0.5 + 4$
 $= 0.25\theta^2 + 0.25\theta + 3.75$
 $df/d\theta = 0.5\theta + 0.25$

stationary point at $\theta = -0.5$, so $(x_2, y_2) = (0.5, 0.25)$.

With shortages:

(b) To minimise costs, differentiate TCU with respect to y.

$$\frac{d}{dy}TCU = \frac{-KD}{y^2} + \frac{h}{2}, \text{ so at S.P.s, } \frac{KD}{y^2} = \frac{h}{2} \Rightarrow y^2 = \frac{2KD}{h} \Rightarrow y^* = \sqrt{2KD/h}$$

as required.

$$TCU(y^*) = \frac{KD}{y^*} + \frac{hy^*}{2} = \frac{KD}{\sqrt{2KD/h}} + \frac{h\sqrt{2KD/h}}{2} = \sqrt{\frac{KDh}{2}} + \sqrt{\frac{KDh}{2}} = \sqrt{2KDh}$$

as required.

With K = 80, D = 360, h = 0.64, then $y^* = \sqrt{2 \times 80 \times 360/0.64} = \sqrt{90000} = 300$ items $TCU(y^*) = \sqrt{2 \times 80 \times 360 \times 0.64} = \sqrt{36864} = \pounds 192$ Average stock held $= y^*/2 = 300/2 = 150$ items $T = y^*/D = 300/360 = 0.83$ weeks.

To find required range of y, set $\rho = y/y^*$, then

$$1.03 = (\rho + (1/\rho))/2$$

$$2.06\rho = \rho^2 + 1$$

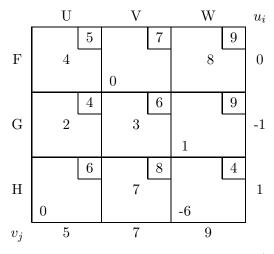
$$\rho^2 - 2.06\rho + 1 = 0$$

$$\rho = (2.06 \pm \sqrt{2.06^2 - 4})/2 = 1.03 \pm 0.2468$$

$$= 0,7832, 1.2768$$

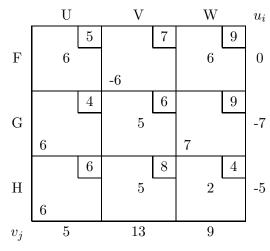
So required range is $[0.7832y^*, 1.2768y^*] = [235, 383]$ items.

.. (a)

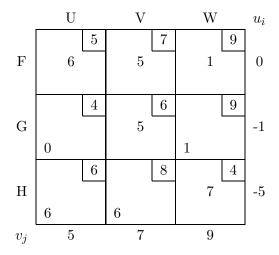


(Compute u_i , v_j using $u_i + v_j = c_{ij}$ for cells ij in the basis (and $u_F = 0$), then for non-basic cells compute $\delta_{ij} = c_{ij} - u_i - v_j$.)

Not optimal, as δ_{HW} is negative. Increase flow through cell with most negative δ value, ie cell HW, by as much as possible.

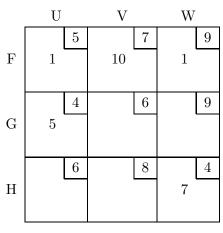


Increase flow through cell FV.

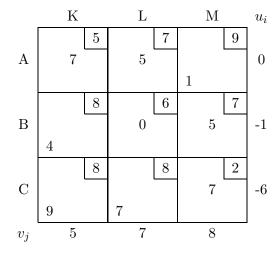


No negative δ values, so optimum has been attained.

gives the alternative optimal solution \mathcal{L}



(b) Initial basic feasible solution is



No negative δ values, so solution is optimal.

If supply at A is increased to 13, then total supply is 25, total demand is 24. To model as a balanced problem, introduce a dummy destination with demand 1, with 'transportation costs' to the dummy destination representing costs of over-production, eg storage costs at the three sources.

If demand at M then increases by 10, no need for dummy destination, total demand is 34, total supply is 25. To model as a balanced problem, need to introduce a dummy source with supply = 9, with 'transportation costs' from the dummy source being used to represent the cost of failing to meet demand.