MATH201 January 2005 Exam Solutions

1. (a) Define

x_1 = Tons of A produced per week

x_2 = Tons of B produced per week

Problem is to maximise $x_0 = 400x_1 + 300x_2$ (£ per week)

subject to

$$\begin{aligned} 0.3x_1 + 0.4x_2 &\leq 300 \text{ (tons per week)} \\ x_1 &\geq 500 \text{ (tons per week)} \\ x_2 &\geq 200 \text{ (tons per week)} \\ x_1, x_2 &\geq 0 \end{aligned}$$

(b) Define

x_1 = Number of drivers working morning shift

x_2 = Number of drivers working split shift

x_3 = Number of drivers working afternoon shift

Problem is to minimise $x_0 = x_1 + x_2 + x_3$ (drivers)

subject to

$$\begin{aligned} x_1 + x_2 &\geq 50 \text{ (drivers)} \\ x_1 &\geq 30 \text{ (drivers)} \\ x_3 &\geq 30 \text{ (drivers)} \\ x_2 + x_3 &\geq 45 \text{ (drivers)} \\ x_3 &\geq 20 \text{ (drivers)} \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Redundant constraints: $x_3 \geq 20$, $x_1 \geq 0$, $x_3 \geq 0$.

2. (a) Feasible region:

Evaluating $z(x, y)$ at vertices of feasible region, $z(2, 0) = 2$, $z(4, 0) = 4$, $z(0, 3) = 12$, $z(0, 2) = 8$, so maximum is $z = 12$ at $(x, y) = (0, 3)$.

At optimality, constraints $x \geq 0$ and $3x + 4y \leq 12$ are binding; $y \geq 0$, $x + y \geq 2$ and $3x + 2y \leq 12$ are non-binding.

Constraint $3x + 2y \leq 12$ is redundant.

(b) Introducing slack variables s_1, s_2, s_3, then tableaux are

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	−3	−1	−1	0	0	0	0
s_1	1	1	1	1	0	0	4
s_2	$\boxed{2}$	1	0	0	1	0	2
s_3	−1	1	1	0	0	1	2

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	0	0.5	−1	0	1.5	0	3
s_1	0	0.5	$\boxed{1}$	1	−0.5	0	3
x_1	1	0.5	0	0	0.5	0	1
s_3	0	1.5	1	0	0.5	1	3

or

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	0	0.5	−1	0	1.5	0	3
s_1	0	0.5	1	1	−0.5	0	3
x_1	1	0.5	0	0	0.5	0	1
s_3	0	1.5	$\boxed{1}$	0	0.5	1	3

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	0	1	0	1	1	0	6
x_3	0	0.5	1	1	−0.5	0	3
x_1	1	0.5	0	0	0.5	0	1
s_3	0	1	0	−1	1	1	0

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	0	2	0	0	2	1	6
s_1	0	−1	0	1	−1	−1	0
x_1	1	0.5	0	0	0.5	0	1
x_3	0	1.5	1	0	0.5	1	3

So optimal solution is $x_0 = 6$, when $x_1 = 1$, $x_2 = 0$, $x_3 = 3$.

Check constraints:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 + 0 + 3 = 4 \leq 4 \\ 2x_1 + x_2 &= 2 + 0 = 2 \leq 2 \\ -x_1 + x_2 + x_3 &= -1 + 0 + 3 = 2 \leq 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Basic variables are x_1, x_3, s_3; alternative optimal basis x_1, x_3, s_1. Or vice-versa. Or could give either x_1, x_3, s_2 or x_1, x_2, x_3 as alternative basis.

3. (a) Feasible region:

Evaluating z at vertices gives $z(0,0) = 0$, $z(2,0) = 4$, $z(0.5,3) = 7$, $z(0,2) = 4$ so maximal value is $z = 7$ at $(x,y) = (0.5,3)$.

(i) Lines $-2x + y = 2$ and $2x = 4$ intersect at $(2,6)$, constraint $2x + y \leq c$ become redundant when line $2x + y = c$ passes through the same point, so when $c = 2 \times 2 + 6 = 10$.

(ii) Optimum remains at $(0.5,3)$ until objective line $2x + by = const$ is parallel to the line $2x + y = 4$, that is when $b = 1$, so within the range $b \geq 1$ the optimum point remains the same.

(iii) Optimal solution is affected when k increases so that the line $2x + ky = 4$ crosses into the feasible region, which happens when it coincides with the line $2x+y = 4$, so when $k = 1$. That is, k can increase by 1 before solution is affected.

(b) Dual simplex method appropriate when there are $\geq$ constraints, and usually for minimisation problems.

Introducing surplus variables s_1, s_2 and slack variable s_3, then tableaux are

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	3	1	1	0	0	0	0
s_1	1	-2	-1	1	0	0	-2
s_2	-2	1	[-1]	0	1	0	-5
s_3	1	1	2	0	0	1	12

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	1	2	0	0	1	0	-5
s_1	3	-3	0	1	-1	0	3
x_3	2	-1	1	0	-1	0	5
s_3	-3	3	0	0	2	1	2

So optimal solution is $x_0 = -5$, when $x_1 = 0$, $x_2 = 0$, $x_3 = 5$.

1. (a) Dual is

D: minimise $y_0 = 7y_1 + y_2 + 2y_3$
subject to

$$\begin{aligned} 2y_1 - y_2 + y_3 &\geq 1 \\ 3y_1 + y_2 &\geq 1 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Given $x_1^* = 2$, $x_2^* = 1$ then
$2x_1^* + 3x_2^* = 7$, first constraint satisfied with equality;
$-x_1^* + x_2^* = -1$, second constraint not satisfied with equality;
$x_1^* = 2$, third constraint satisfied with equality.
Given $y_1^* = \frac{1}{3}$, $y_2^* = 0$, $y_3^* = \frac{1}{3}$ then
$2y_1^* - y_2^* + y_3^* = 1$, first constraint satisfied with equality;
$3y_1^* + y_2^* = 1$, second constraint satisfied with equality.
Complementary slackness requires that for those constraints not satisfied with equality, the dual variable is zero. For the primal, the second constraint is the only one not satisfied with equality, so since $y_2^* = 0$, complementary slackness is satisfied. For the dual, both constraints are satisfied with equality, so complementary slackness is satisfied.

(b) O is inferior to A, B, C, D; D is inferior to B, C.
The NIS consists of the edges AB and BC.
When $w = 0$, optimal point is A.
When $w = 1$, optimal point is C.
Optimum moves from A to B when

$$\begin{aligned} 19(1-w) + 2w &= 14(1-w) + 16w \\ 19 - 17w &= 14 + 2w \\ -19w &= -5 \\ w &= 5/19 \end{aligned}$$

Optimum moves from B to C when

$$\begin{aligned} 14(1-w) + 16w &= 10(1-w) + 17w \\ 14 + 2w &= 10 + 7w \\ -5w &= -4 \\ w &= 4/5 \end{aligned}$$

$0 \leq w < 5/19$: point A optimal
$w = 5/19$: edge AB optimal
$5/19 < w < 4/5$: point B optimal
$w = 4/5$: edge BC optimal
$4/5 < w \leq 1$: point C optimal

5. (a) Lagrangean is $L(x, y, \lambda) = x^2 + y^2 + 3xy + 5x + 10y + \lambda(4x + y - 5)$, so for optimum require

$$\begin{aligned} L_x &= 2x + 3y + 5 + 4\lambda &= 0 \\ L_y &= 2y + 3x + 10 + \lambda &= 0 \\ L_\lambda &= 4x + y - 5 &= 0 \end{aligned}$$

To solve for x, y, λ,

$$3L_x - 2L_y = 5y - 5 + 10\lambda = 0 \Rightarrow y = 1 - 2\lambda$$
$$\text{then } L_x = 0 \Rightarrow x = -(3y + 5 + 4\lambda)/2 = -(3 - 6\lambda + 5 + 4\lambda)/2 = -(8 - 2\lambda)/2 = -4 + \lambda$$
$$\text{next, } L_\lambda = 0 \Rightarrow 4(-4 + \lambda) + (1 - 2\lambda) - 5 = 0 \Rightarrow 2\lambda - 20 = 0 \Rightarrow \lambda = 10$$
$$\text{hence minimum occurs at } x^* = 6, y^* = -19.$$

(b)

$$\begin{aligned} f(x, y) &= x^2 + y^2 - xy - x + 4 \\ \nabla f &= (2x - y - 1,\ 2y - x) \end{aligned}$$

Starting from $(x_0, y_0) = (0, 0)$, have $(\nabla f)_0 = (-1, 0)$, so search along the line

$$\begin{aligned} (x, y) &= (0, 0) + \theta(-1, 0) &= (-\theta, 0) \\ \text{so } f(x, y) &= \theta^2 + \theta + 4 \\ df/d\theta &= 2\theta + 1 \end{aligned}$$

stationary point at $\theta = -0.5$, so $(x_1, y_1) = (0.5, 0)$.
Hence $(\nabla f)_1 = (0, -0.5)$, so search along the line

$$\begin{aligned} (x, y) &= (0.5, 0) + \theta(0, -0.5) &= (0.5, -0.5\theta) \\ \text{so } f(x, y) &= 0.25 + 0.25\theta^2 + 0.25\theta - 0.5 + 4 \\ &= 0.25\theta^2 + 0.25\theta + 3.75 \\ df/d\theta &= 0.5\theta + 0.25 \end{aligned}$$

stationary point at $\theta = -0.5$, so $(x_2, y_2) = (0.5, 0.25)$.

6. (a) Stock level against time.

With shortages:

(b) To minimise costs, differentiate TCU with respect to y.

$$\frac{d}{dy}TCU = \frac{-KD}{y^2} + \frac{h}{2}, \text{ so at S.P.s, } \frac{KD}{y^2} = \frac{h}{2} \Rightarrow y^2 = \frac{2KD}{h} \Rightarrow y^* = \sqrt{2KD/h}$$

as required.

$$TCU\left(y^*\right) = \frac{KD}{y^*} + \frac{hy^*}{2} = \frac{KD}{\sqrt{2KD/h}} + \frac{h\sqrt{2KD/h}}{2} = \sqrt{\frac{KDh}{2}} + \sqrt{\frac{KDh}{2}} = \sqrt{2KDh}$$

as required.

With $K = 80,\ D = 360,\ h = 0.64$, then

$y^* = \sqrt{2 \times 80 \times 360/0.64} = \sqrt{90000} = 300$ items

$TCU\left(y^*\right) = \sqrt{2 \times 80 \times 360 \times 0.64} = \sqrt{36864} = £192$

Average stock held $= y^*/2 = 300/2 = 150$ items

$T = y^*/D = 300/360 = 0.83$ weeks.

To find required range of y, set $\rho = y/y^*$, then

$$\begin{aligned} 1.03 &= (\rho + (1/\rho))/2 \\ 2.06\rho &= \rho^2 + 1 \\ \rho^2 - 2.06\rho + 1 &= 0 \\ \rho &= \left(2.06 \pm \sqrt{2.06^2 - 4}\right)/2 = 1.03 \pm 0.2468 \\ &= 0, 7832, 1.2768 \end{aligned}$$

So required range is $[0.7832y^*, 1.2768y^*] = [235, 383]$ items.

7. (a)

	U	V	W	u_i
F	[5] 4	[7] 0	[9] 8	0
G	[4] 2	[6] 3	[9] 1	-1
H	[6] 0	[8] 7	[4] -6	1
v_j	5	7	9	

(Compute u_i, v_j using $u_i + v_j = c_{ij}$ for cells ij in the basis (and $u_F = 0$), then for non-basic cells compute $\delta_{ij} = c_{ij} - u_i - v_j$.)

Not optimal, as δ_{HW} is negative. Increase flow through cell with most negative δ value, ie cell HW, by as much as possible.

	U	V	W	u_i
F	[5] 6	[7] -6	[9] 6	0
G	[4] 6	[6] 5	[9] 7	-7
H	[6] 6	[8] 5	[4] 2	-5
v_j	5	13	9	

Increase flow through cell FV.

	U	V	W	u_i
F	[5] 6	[7] 5	[9] 1	0
G	[4] 0	[6] 5	[9] 1	-1
H	[6] 6	[8] 6	[4] 7	-5
v_j	5	7	9	

No negative δ values, so optimum has been attained.

Have $\delta_{GU} = 0$, so there is an alternative optimal basis. Bringing GU into the basis gives the alternative optimal solution

	U	V	W
F	1 [5]	10 [7]	1 [9]
G	5 [4]	[6]	[9]
H	[6]	[8]	7 [4]

(b) Initial basic feasible solution is

	K	L	M	u_i
A	7 [5]	5 [7]	[9] (1)	0
B	[8] (4)	0 [6]	5 [7]	-1
C	[8] (9)	[8] (7)	7 [2]	-6
v_j	5	7	8	

No negative δ values, so solution is optimal.

If supply at A is increased to 13, then total supply is 25, total demand is 24. To model as a balanced problem, introduce a dummy destination with demand 1, with 'transportation costs' to the dummy destination representing costs of over-production, eg storage costs at the three sources.

If demand at M then increases by 10, no need for dummy destination, total demand is 34, total supply is 25. To model as a balanced problem, need to introduce a dummy source with supply = 9, with 'transportation costs' from the dummy source being used to represent the cost of failing to meet demand.