MATH261/761 January 2002 Exam Solutions

1. (a) Define

   $x_1 =$ Tons of exterior paint produced per day

   $x_2 =$ Tons of interior paint produced per day

   Problem is to maximise $x_0 = 3000x_1 + 2000x_2$ (£ per day) subject to

$$\begin{aligned} 0.3x_1 + 0.6x_2 &\leq 16 \text{ (tons per day)} \\ 0.7x_1 + 0.4x_2 &\leq 12 \text{ (tons per day)} \\ -x_1 + x_2 &\leq 2 \text{ (tons per day)} \\ x_2 &\leq 5 \text{ (tons per day)} \\ x_1, x_2 &\geq 0 \end{aligned}$$

   (b) Feasible region:

   Evaluate x_0 at vertices of feasible region: $x_0(1,0) = 1$, $x_0(2,0) = 2$, $x_0(4,1) = 6$, $x_0(2,3) = 8$, $x_0(1,2) = 5$. Hence optimal solution is $x_0 = 8$ at $(2,3)$.

   Redundant constraints are $x_1 + 2x_2 \leq 9$, $x_2 \leq 3$ and $x_1 \geq 0$.

   [**Y3 version:**

   Optimal point (x_1, x_2) changes when objective line is parallel to $x_1 + x_2 = 5$ or $-x_1 + x_2 = 1$; that is, when $a = 2$ or $a = -2$. So the required range of a values is $-2 \leq a \leq 2$.]

2. [**Y3 version:**

(a) First choose column with most negative objective row entry as the pivot column. Then, for each positive entry in the pivot column, find the ratio between the entry in the same row in the constants column and the entry in the pivot column. The row for which this ratio is smallest is the pivot row.

The purpose of the ratio test is to ensure that when the pivot operation is carried out, no negative entries are introduced into the constants column, since this would mean that one of the variables took a negative value, and so would violate the non-negativity constraints.

]

(a) Introducing slack variables s_1, s_2, s_3, then tableaux are

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	-2	-3	-5	0	0	0	0
s_1	1	1	-1	1	0	0	5
s_2	6	-2	-9	0	1	0	4
s_3	1	1	$\boxed{4}$	0	0	1	10

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	-0.75	-1.75	0	0	0	1.25	12.5
s_1	1.25	$\boxed{1.25}$	0	1	0	0.25	7.5
s_2	8.25	0.25	0	0	1	2.25	26.5
x_3	0.25	0.25	1	0	0	0.25	2.5

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	1	0	0	1.4	0	1.6	23
x_2	1	1	0	0.8	0	0.2	6
s_2	8	0	0	-0.2	1	2.2	25
x_3	0	0	1	-0.2	0	0.2	1

So optimal solution is $x_0 = 23$, when $x_1 = 0$, $x_2 = 6$, $x_3 = 1$.

(b) Introducing surplus variable s and artificial variables A_1, A_2, constraints become

$$\begin{aligned} x_1 + x_2 - s + A_1 &= 4 \\ 2x_1 + 5x_3 + A_2 &= 10 \\ x_1, x_2, x_3, s, A_1, A_2 &\geq 0 \end{aligned}$$

and objective becomes

$$\begin{aligned} x_0 &= -3x_1 - x_2 + 3x_3 - M(A_1 + A_2) \\ &= -3x_1 - x_2 + 3x_3 - M(4 - x_1 - x_2 + s + 10 - 2x_1 - 5x_3) \\ &= (-3 + 3M)x_1 + (-1 + M)x_2 + (3 + 5M)x_3 - Ms - 14M \end{aligned}$$

so that tableaux are

	x_1	x_2	x_3	s	A_1	A_2	
x_0	$3 - 3M$	$1 - M$	$-3 - 5M$	M	0	0	$-14M$
A_1	1	1	0	-1	1	0	4
A_2	2	0	$\boxed{5}$	0	0	1	10

	x_1	x_2	x_3	s	A_1	A_2	
x_0	$4.2-M$	$1-M$	0	M	0	·	$6-4M$
A_1	1	$\boxed{1}$	0	-1	1	·	4
x_3	0.4	0	1	0	0	·	2

	x_1	x_2	x_3	s	A_1	A_2	
x_0	3.2	0	0	1	·	·	2
x_2	1	1	0	-1	·	·	4
x_3	0.4	0	1	0	·	·	2

So optimal solution is $x_0 = 2$, when $x_1 = 0$, $x_2 = 4$, $x_3 = 2$.

3. (a) Dual is
D: minimise $y_0 = 6y_1 + 4y_2$
subject to

$$\begin{aligned} y_1 + 2y_2 &\geq 2 \\ 2y_1 + y_2 &\geq 2 \\ 3y_1 + 3y_2 &\geq 3 \\ y_1, y_2 &\geq 0 \end{aligned}$$

Dual feasible region:

Evaluating y_0 at vertices gives $y_0(2,0) = 12$, $y_0(0,2) = 8$, $y_0(2/3, 2/3) = 20/3$, so minimal value is $y_0 = 20/3$ at $(2/3, 2/3)$.

Hence maximal value of x_0 for primal problem is $x_0 = 20/3$.

[**Y3 version:**

Complementary slackness conditions say that at optimal points,

$$\begin{aligned} x_1\,(y_1 + 2y_2 - 2) &= 0 \\ x_2\,(2y_1 + y_2 - 2) &= 0 \end{aligned}$$

$$\begin{aligned} x_3\,(3y_1 + 3y_2 - 3) &= 0 \\ y_1\,(x_1 + 2x_2 + 3x_3 - 6) &= 0 \\ y_2\,(2x_1 + x_2 + 3x_3 - 4) &= 0 \end{aligned}$$

Substituting in $y_1 = 2/3$, $y_2 = 2/3$ the first three conditions become

$$\begin{aligned} x_1 \times 0 &= 0 \\ x_2 \times 0 &= 0 \\ x_3 \times 1 &= 0 \end{aligned}$$

Putting $x_3 = 0$ into the last two conditions, then

$$\begin{aligned} x_1 + 2x_2 &= 6 \\ 2x_1 + x_2 &= 4 \end{aligned}$$

so that $x_1 = 2/3$, $x_2 = 8/3$, $x_3 = 0$.]

(b) Introducing surplus variables s_1, s_2, s_3, then tableaux are

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	4	3	3	0	0	0	0
s_1	-1	-1	$\boxed{-1}$	1	0	0	-10
s_2	-1	-2	-2	0	1	0	-4
s_3	-1	1	2	0	0	1	-4

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	1	0	0	3	0	0	-30
x_3	1	1	1	-1	0	0	10
s_2	1	0	0	-2	1	0	16
s_3	-3	$\boxed{-1}$	0	2	0	1	-24

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	1	0	0	3	0	0	-30
x_3	$\boxed{-2}$	0	1	1	0	1	-14
s_2	1	0	0	-2	1	0	16
x_2	3	1	0	-2	0	-1	24

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	0	0	0.5	3.5	0	0.5	-37
x_1	1	0	-0.5	-0.5	0	-0.5	7
s_2	0	0	0.5	-1.5	1	0.5	9
x_2	0	1	1.5	-0.5	0	0.5	3

So optimal solution is $x_0 = -37$, when $x_1 = 7$, $x_2 = 3$, $x_3 = 0$.

4. [**Y3 version:**

(a) The weighting method is to maximise $(1-w)Z_1 + wZ_2$ for $0 \le w \le 1$. Used because it is usually not possible to simultaneously maximise both Z_1 and Z_2, so instead maximise some weighted average of the two. Partial information about the relative importance of Z_1 and Z_2 may then be enough to determine which values of the decision variables to use; for instance, if we know that Z_1 is more important than Z_2, this means that $w < 1/2$, and it may happen that the optimal values of the decision variables are the same for all such w values.]

(i)

(ii) $Z_1 = x + 2y$, $Z_2 = 3x + y$.

	x	y	Z_1	Z_2
A	1	0	1	3
B	1	7	15	10
C	3	5	13	14
D	4	3	10	15
E	2	0	2	6

A is inferior to B, and E is inferior to D.

The NIS consists of the edges BC and CD.

(iii) When $w = 0$, optimal point is B.

When $w = 1$, optimal point is D.

Optimum moves from B to C when

$$\begin{aligned} (1-w)Z_1(B) + wZ_2(B) &= (1-w)Z_1(C) + wZ_2(C) \\ 15(1-w) + 10w &= 13(1-w) + 14w \\ 15 - 5w &= 13 + w \end{aligned}$$

$$
\begin{aligned}
6w &= 2 \\
w &= 1/3
\end{aligned}
$$

Optimum moves from C to D when

$$
\begin{aligned}
(1-w)Z_1(C) + wZ_2(C) &= (1-w)Z_1(D) + wZ_2(D) \\
13(1-w) + 14w &= 10(1-w) + 15w \\
13 + w &= 10 + 5w \\
4w &= 3 \\
w &= 3/4
\end{aligned}
$$

$0 \leq w < 1/3$: point B optimal
$w = 1/3$: edge BC optimal
$1/3 < w < 3/4$: point C optimal
$w = 3/4$: edge CD optimal
$3/4 < w \leq 1$: point D optimal

(iv)

(v) With x, y integers, set of feasible points consists of the five vertices together with

x	y	Z_1	Z_2
1	1	3	4
1	2	5	5
1	3	7	6
1	4	9	7
1	5	11	8
1	6	13	9
2	1	4	7
2	2	6	8
2	3	8	9
2	4	10	10
2	5	12	11
2	6	14	12
3	2	7	11
3	3	9	12
3	4	11	13

Points with $x = 1$ are all inferior to point B $= (1, 7)$.

Points with $x = 2$ are all inferior to point $(2, 6)$.

Points with $x = 3$ are all inferior to point C $= (3, 5)$.

NIS is $\{(1, 7), (2, 6), (3, 5), (4, 3)\}$.

5. (a) With $K = 80$, $D = 100$, $h = 0.03$, then $y^* = \sqrt{2 \times 80 \times 100/0.03} \approx 730.3$, $TCU\,(y^*) = (80 \times 100/730.3) + (0.03 \times 730.3/2) \approx £21.91$.

Orders in multiples of 50 units:

$$\begin{aligned} TCU(700) &= (80 \times 100/700) + (0.03 \times 700/2) \approx 21.93 \\ TCU(750) &= (80 \times 100/750) + (0.03 \times 750/2) \approx 21.92 \end{aligned}$$

So minimal cost is achieved by an order size of 750.

If the working week is 7 days long, then would recommend an order size of 700, since then orders are placed once each week, and the increase in costs compared to the minimal cost solution is negligible.

(b) Stock level against time:

From the graph,

$$\begin{aligned}
\text{Cost per cycle} &= K + \frac{1}{2}t_1(y-w)h + \frac{1}{2}t_2 wp \\
\text{and } \frac{w}{t_2} = \frac{y-w}{t_1} = \frac{y}{t_1+t_2} = D & \\
\text{so that Cost per cycle} &= K + \frac{h(y-w)^2}{2D} + \frac{pw^2}{2D} \\
\text{and hence } TCU(y,w) &= \frac{\text{Cost per cycle}}{t_1+t_2} \\
&= \text{Cost per cycle} \times \frac{D}{y} \\
&= \frac{KD}{y} + \frac{h(y-w)^2}{2y} + \frac{pw^2}{2y}
\end{aligned}$$

as required.

To minimise costs, differentiate TCU with respect to each of y and w.

$$\begin{aligned}
TCU(y,w) &= \frac{KD}{y} + \frac{hy}{2} - hw + \frac{hw^2}{2y} + \frac{pw^2}{2y}, \\
\frac{d}{dw}TCU &= -h + \frac{hw}{y} + \frac{pw}{y}, \\
\frac{d}{dw}TCU = 0 \Rightarrow \frac{(h+p)w}{y} &= h \\
\Rightarrow \frac{w}{y} &= \frac{h}{h+p}. \\
\frac{d}{dy}TCU &= \frac{-KD}{y^2} + \frac{h}{2} - \frac{(h+p)w^2}{2y^2}, \\
\frac{d}{dy}TCU = 0 \Rightarrow \frac{KD}{y^2} + \frac{(h+p)w^2}{2y^2} &= \frac{h}{2}, \\
\frac{w}{y} = \frac{h}{h+p} \Rightarrow \frac{KD}{y^2} + \frac{(h+p)h^2}{2(h+p)^2} &= \frac{h}{2} \\
\Rightarrow \frac{KD}{y^2} &= \frac{h}{2} - \frac{h^2}{2(h+p)} \\
&= \frac{h(h+p)-h^2}{2(h+p)}
\end{aligned}$$

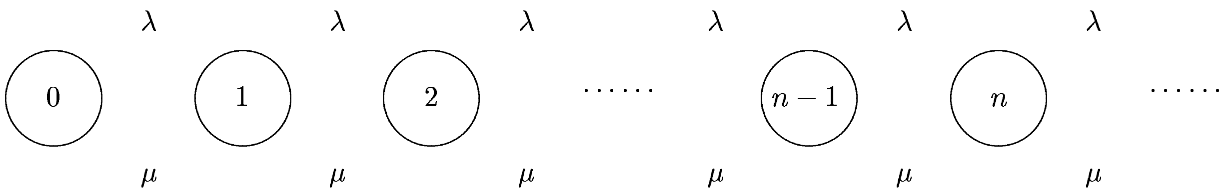
$$
\begin{aligned}
&= \frac{hp}{2(h+p)} \\
\Rightarrow y^2 &= \frac{2KD(h+p)}{hp} \\
\Rightarrow y &= \sqrt{\frac{2KD(h+p)}{hp}} \\
\text{and hence } w &= \frac{h}{h+p}\sqrt{\frac{2KD(h+p)}{hp}} = \sqrt{\frac{2KDh}{(h+p)p}}
\end{aligned}
$$

[**Y3 version:**

As $p \to \infty$, then minimising values of y and w converge to $\sqrt{2KD/h}$ and 0, respectively. That is, as shortage costs become very high, it becomes optimal never to allow shortages to occur, and so the optimal y value is as in part (a) of the question, when shortages weren't permitted.

As $p \to 0$, then minimising values of y and w both tend to infinity, with $y/w \to 1$. That is, if shortages cost very little, then it becomes optimal to allow very large shortages to occur, with the maximum stock level $y - w$ becoming very small in comparison to w.]

6. (a) State transition diagram for $(M/M/1)$ queue:



The $(M/M/2)$ queue differs in that there are two servers, operating independently. Thus if each server operates at mean rate μ, then the total service rate is 2μ, except when there's only one customer in the system, service rate then being μ.

(b) Balancing probability flows, and recalling $\rho = \lambda/\mu$,

$$
\begin{aligned}
\lambda p_0 &= \mu p_1 & p_1 &= \rho p_0 \\
(\lambda/2) p_1 &= \mu p_2 & p_2 &= (\rho/2) p_1 = (\rho^2/2)\, p_0 \\
(\lambda/3) p_2 &= \mu p_3 & p_3 &= (\rho/3) p_2 = (\rho^3/6)\, p_0 \\
&\vdots & &\vdots \\
(\lambda/n) p_{n-1} &= \mu p_n & p_n &= (\rho/n)\, p_{n-1} = (\rho^n/n!)\, p_0 \\
&\vdots & &\vdots
\end{aligned}
$$

Probabilities sum to 1, so normalising gives

$$
\begin{aligned}
p_0 + p_1 + p_2 + \cdots &= 1 \\
p_0 + \rho p_0 + \left(\rho^2/2\right) p_0 + \left(\rho^3/6\right) p_0 + \cdots &= 1
\end{aligned}
$$

$$
\begin{aligned}
p_0\left(1+\rho+\left(\rho^2/2\right)+\left(\rho^3/6\right)+\cdots\right) &= 1 \\
p_0\exp(\rho) &= 1 \\
p_0 &= \exp(-\rho)
\end{aligned}
$$

So for $n = 1, 2, \ldots,$

$$
p_n = (\rho^n/n!)\,p_0 = \frac{\rho^n\exp(-\rho)}{n!}
$$

(c) (i) Balancing probability flows,

$$
\begin{aligned}
2\alpha p_0 &= \beta p_1 \\
2\alpha p_1 &= 2\beta p_2 \\
\alpha p_2 &= 3\beta p_3
\end{aligned}
$$

So with $\rho = \alpha/\beta = 3/4 = 0.75$ then $p_1 = 2\rho p_0$, $p_2 = 2\rho^2 p_0$, $p_3 = (2/3)\rho^3 p_0$. Thus $p_0\left(1+2\rho+2\rho^2+(2/3)\rho^3\right) = 1$, and

$$
\begin{aligned}
p_0 &= \frac{1}{1+2\rho+2\rho^2+(2/3)\rho^3} = 1/3.90625 = 0.256 \\
p_1 &= 2\times 0.75\times p_0 = 0.384 \\
p_2 &= 2\times 0.75^2\times p_0 = 0.288 \\
p_3 &= (2/3)\times 0.75^3\times p_0 = 0.072
\end{aligned}
$$

(ii) Average number of machines in operation $= p_1 + 2p_2 + 3p_3 = 0.384 + 2\times 0.288 + 3\times 0.072 = 1.176$

[**Y3 version:**

(iii) Average time with all out of operation $= 8\times p_0 = 2.048$ hours.]

7. (a)

	1	2	3	u_i
A	$\boxed{550}$ 15	$\boxed{575}$ 5	$\boxed{520}$ -70	0
B	$\boxed{530}$ 5	$\boxed{550}$ 17	$\boxed{565}$ -5	-20
C	$\boxed{540}$ 40	$\boxed{520}$ 3	$\boxed{540}$ 15	-50
D	$\boxed{550}$ 30	$\boxed{560}$ 20	$\boxed{560}$ 9	-30
v_j	550	570	590	

(Compute u_i, v_j using $u_i + v_j = c_{ij}$ for cells ij in the basis (and $u_A = 0$), then for non-basic cells compute $\delta_{ij} = c_{ij} - u_i - v_j$.)

Not optimal, as δ_{A3} and δ_{B3} are negative.

If demand at A rises to 24, then total demand is 68, total supply is 64. To model as a balanced problem, need to introduce a dummy source supplying 4 thousand items per week, with 'transportation costs' from the dummy source being used to represent the cost of failing to meet demand.

(b) Compute u_i, v_j using $u_i + v_j = c_{ij}$ for cells ij in the basis (and $u_U = 0$), then for non-basic cells compute $\delta_{ij} = c_{ij} - u_i - v_j$.

	A	B	C	D	u_i
U	$\boxed{6}$ 10	$\boxed{4}$ 5	$\boxed{6}$ 14	$\boxed{15}$ 18	0
V	$\boxed{12}$ -10	$\boxed{20}$ 4	$\boxed{8}$ 11	$\boxed{4}$ -9	16
W	$\boxed{10}$ -9	$\boxed{4}$ -13	$\boxed{5}$ 12	$\boxed{10}$ 3	13
v_j	6	4	-8	-3	

Increase flow through cell with most negative δ value, ie cell WB, by as much as possible.

	A	B	C	D	u_i
U	$\boxed{6}$ 10	$\boxed{4}$ 5	$\boxed{6}$ 1	$\boxed{15}$ 5	0
V	$\boxed{12}$ 3	$\boxed{20}$ 13	$\boxed{8}$ 15	$\boxed{4}$ -9	3
W	$\boxed{10}$ 4	$\boxed{4}$ 4	$\boxed{5}$ 8	$\boxed{10}$ 3	0
v_j	6	4	5	10	

Most negative δ value now in cell VD, so increase flow there.

	A	B	C	D	u_i
U	$\boxed{6}$ 10	$\boxed{4}$ 5	$\boxed{6}$ 1	$\boxed{15}$ 14	0
V	$\boxed{12}$ 3	$\boxed{20}$ 13	$\boxed{8}$ 12	$\boxed{4}$ 3	3
W	$\boxed{10}$ 4	$\boxed{4}$ 4	$\boxed{5}$ 11	$\boxed{10}$ 9	0
v_j	6	4	5	1	

No negative δ values, so optimum has been attained.

[**Y3 version:**

	A	B	C	D	u_i
U	$\boxed{6}$ 10	$\boxed{4}$ 5	$\boxed{5}$ 13	$\boxed{15}$ 18	0
V	$\boxed{12}$ -10	$\boxed{20}$ 4	$\boxed{8}$ 11	$\boxed{4}$ -9	16
W	$\boxed{10}$ -9	$\boxed{4}$ -13	$\boxed{5}$ 12	$\boxed{10}$ 3	13
v_j	6	4	-8	-3	

Increase flow through cell with most negative δ value, ie cell WB, by as much as possible.

	A	B	C	D	u_i
U	$\boxed{6}$ 10	$\boxed{4}$ 5	$\boxed{5}$ 0	$\boxed{15}$ 5	0
V	$\boxed{12}$ 3	$\boxed{20}$ 13	$\boxed{8}$ 15	$\boxed{4}$ -9	3
W	$\boxed{10}$ 4	$\boxed{4}$ 4	$\boxed{5}$ 8	$\boxed{10}$ 3	0
v_j	6	4	5	10	

Most negative δ value now in cell VD, so increase flow there.

	A	B	C	D	u_i
U	$\boxed{6}$ 10	$\boxed{4}$ 5	$\boxed{5}$ 0	$\boxed{15}$ 14	0
V	$\boxed{12}$ 3	$\boxed{20}$ 13	$\boxed{8}$ 12	$\boxed{4}$ 3	3
W	$\boxed{10}$ 4	$\boxed{4}$ 4	$\boxed{5}$ 11	$\boxed{10}$ 9	0
v_j	6	4	5	1	

No negative δ values, so optimum has been attained.

Cost of initial solution $= 6 \times 10 + 4 \times 5 + 20 \times 4 + 8 \times 11 + 5 \times 12 + 10 \times 3 = 338$.

Cost of optimal solution $= 6 \times 10 + 4 \times 5 + 8 \times 12 + 4 \times 3 + 4 \times 4 + 5 \times 11 = 259$.

Since $\delta_{UC} = 0$, there is an alternative optimal solution, obtained by increasing the flow in cell UC.

	A	B	C	D
U	$\boxed{6}$ 10	$\boxed{4}$	$\boxed{5}$ 5	$\boxed{15}$
V	$\boxed{12}$	$\boxed{20}$	$\boxed{8}$ 12	$\boxed{4}$ 3
W	$\boxed{10}$	$\boxed{4}$ 9	$\boxed{5}$ 6	$\boxed{10}$

]